PLURICLOSED FLOW, BORN-INFELD GEOMETRY, AND RIGIDITY RESULTS FOR GENERALIZED KÄHLER MANIFOLDS

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Abstract. We prove long time existence and convergence results for the pluriclosed flow, which imply geometric and topological classification theorems for generalized Kähler structures. Our approach centers on the reduction of pluriclosed flow to a degenerate parabolic equation for a $(1,0)$-form, introduced in [37]. We observe a number of differential inequalities satisfied by this system which lead to a priori L^{∞} estimates for the metric along the flow. Moreover we observe an unexpected connection to "Born-Infeld geometry" which leads to a sharp differential inequality which can be used to derive an Evans-Krylov type estimate for the degenerate parabolic system of equations. To show convergence of the flow we generalize Yau's oscillation estimate to the setting of generalized Kähler geometry.

1. Introduction

1.1. **Global existence on negatively curved backgrounds.** Given (M^{2n}, g, J) a Hermitian manifold, we say that the metric is *pluriclosed* if

$$\partial\overline{\partial}\omega = 0.$$

Now consider the *pluriclosed flow* equation

$$\frac{\partial}{\partial t}\omega = \partial\partial_g^*\omega + \overline{\partial\partial_g^*}\omega + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g. \tag{1.1}$$

This equation was introduced in [36] as a natural geometric flow on complex manifolds which preserves pluriclosed metrics. In [37] it was established that this flow is the gradient flow of a Perelman-type quantity, and a conjectural framework was established for understanding the singularity formation of solutions to this flow. Further works [6, 9, 13] on pluriclosed flow have also appeared.

Our goal in this paper is to present global existence and convergence results for this flow and the geometric and topological rigidity results which follow as corollaries. The first principal result shows long time existence and convergence for the pluriclosed flow with arbitrary initial data on certain complex manifolds.

Theorem 1.1. *Let (M^{2n}, h, J) be a compact Hermitian manifold.*

(1) *If h has nonpositive bisectional curvature, then the solution to (normalized) pluriclosed flow with arbitrary initial data exists smoothly on $[0, \infty)$.*
(2) *If (M^{2n}, J) is biholomorphic to a torus, the solution to pluriclosed flow with arbitrary initial data converges exponentially to a flat Kähler metric.*
(3) *If h has constant negative bisectional curvature, the solution to normalized pluriclosed flow with arbitrary initial data exists for all time and converges exponentially to g_{KE}, the unique Kähler-Einstein metric on (M^{2n}, J).*

This theorem confirms the intuition that the pluriclosed flow does not develop "local" singularities akin to the Ricci flow neckpinch, as in principle one would see these for instance in the case of the

Date: February 7th, 2015.
J. Streets gratefully acknowledges support from the NSF via DMS-1301864, and from an Alfred P. Sloan Fellowship.

torus. Rather, this suggests that the singularities are based on topological obstructions, as in the case of Kähler-Ricci flow. Moreover we point out that the convergence to flat metrics on the tori stands in contrast to the parabolic Monge-Ampere equation in this setting, which always converges ([17]), but admits an infinite dimensional space of fixed points with arbitrary torsion tensor.

1.2. **Rigidity of generalized Kähler manifolds via pluriclosed flow.** We also prove a new long time existence and convergence result for the pluriclosed flow in the setting of commuting generalized Kähler geometry. A generalized Kähler structure on a compact manifold M is a triple (g, J_A, J_B) of a Riemannian metric and integrable complex structures so that

$$\begin{aligned} d^c_A \omega_A =& \ - d^c_B \omega_B \\ dd^c_A \omega_A =& \ - dd^c_B \omega_B = 0. \end{aligned}$$

These equations first arose in the context of supersymmetric sigma models [15]. Later these equations were given a purely geometric interpretation in the language of Hitchin's "generalized geometry," [18, 19, 21].

In [38] the author and Tian showed that the pluriclosed flow preserves generalized Kähler structure, provided we allow the complex structures to evolve by diffeomorphisms. Later, in [35] we showed that in the special case when $[J_A, J_B] = 0$, the complex structures remain fixed and moreover the flow reduces to a fully nonlinear scalar PDE. Moreover, in [35] we gave a nearly complete description of the long time behavior of the pluriclosed flow in the setting when $n = 2$. The reason for the dimensional restriction stems largely from the fact that when $n = 2$ the factors in the splitting of the tangent bundle arising from the commuting complex structures (see §7.1) are both line bundles.

In this paper we develop a number of new a priori estimates for the pluriclosed flow in this setting, leading to a general long time existence and convergence result given certain topological constraints. This leads to a rigidity result showing that under certain topological conditions generalized Kähler structures are automatically covered by products of Calabi-Yau manifolds. First we state a general long time existence result (see §7 for the definition of $\chi(J_A, J_B)$, which is a characteristic class associated to a generalized Kähler manifold akin to the first Chern class of a Kähler manifold).

Theorem 1.2. *Let (M^{2n}, g, J_A, J_B) be a commuting generalized Kähler manifold satisfying the conditions $c_1^{BC}(J_A) \leq 0$, $\chi(J_A, J_B) = 0$, and $\operatorname{rank} T^{1,0}_- = 1$. The solution to pluriclosed flow with initial condition g exists smoothly on $[0, \infty)$.*

The simplest example of a manifold satisfying the hypotheses of Theorem 1.2 is to let (N_i, J_i) be Calabi-Yau manifolds with $\dim N_1 = 1$, and let $M = N_1 \times N_2$, $J_A = J_1 \oplus J_2$, $J_B = J_1 \oplus (-J_2)$. With one further hypothesis on the background complex manifolds, namely that the torsion class $[\partial \omega_A]$ vanishes, we can use the pluriclosed flow to show that, up to coverings, this is the *only* way to construct such manifolds. This follows by establishing convergence of the flow at infinity to a Kähler-Einstein metric. One important input to obtain this convergence is a generalization of Yau's oscillation estimate for the Kähler potential [47] to the setting of generalized Kähler geometry (Theorem 7.7). Note that in the theorem below we do not assume that either of the given complex structures J_A, J_B admits Kähler metrics, rather, this is a consequence.

Theorem 1.3. *Let (M^{2n}, g, J_A, J_B) be a commuting generalized Kähler manifold such that $c_1^{BC}(J_A) = 0$, $\chi(J_A, J_B) = 0$, $\operatorname{rank} T^{1,0}_- = 1$, and $[\partial \omega_A] = 0 \in H^{2,1}$. The solution to pluriclosed flow with initial condition g exists smoothly on $[0, \infty)$ and converges exponentially as $t \to \infty$ to a Calabi-Yau metric. In particular, $(M^{2n}, J_{A,B})$ are both Calabi-Yau manifolds, and the universal cover of (M, J_A, J_B) is biholomorphic to a product manifold.*

We can use this theorem to state a clearer corollary, which says that any rank 1 commuting generalized Kähler structure for which the relevant characteristic classes vanish is automatically Calabi-Yau, and moreover covered by a product structure.

Corollary 1.4. *Let (M^{2n}, g, J_A, J_B) be a commuting generalized Kähler manifold such that $c_1^{BC}(T_{\pm}^{1,0}) = 0$, $\operatorname{rank} T_{-}^{1,0} = 1$, and $[\partial \omega_A] = 0 \in H^{2,1}$. Then $(M, J_{A,B})$ are Calabi-Yau manifolds, and the universal cover of (M, J_A, J_B) is biholomorphic to a product manifold.*

Remark 1.5. We emphasize that the hypotheses of Theorem 1.2, Theorem 1.3 and Corollary 1.4 are all topological in nature. In particular, locally, there is an infinite dimensional family of rank 1 commuting generalized Kähler manifolds. By adding global, topological hypotheses we are able to show the existence of rigid metrics on these manifolds via the pluriclosed flow.

Remark 1.6. The standard generalized Kähler structure on the Hopf surface satisfies $\chi(J_A, J_B) = 0$ and $\operatorname{rank} T_{-}^{1,0} = 1$. Thus we see that to obtain Kähler rigidity, some sort of extra hypothesis is necessary. The Hopf surface example satisfies both $[\partial \omega_A] \neq 0$ and $c_1^{BC}(J_A) \neq 0$. It is not clear yet if either of the hypotheses $[\partial \omega_A] = 0$ or $c_1^{BC}(J_A)$ can be removed while keeping rigidity.

1.3. **A priori estimates.** Proving these theorems requires the development of several new a priori estimates for the pluriclosed flow, which we now outline. A pluriclosed metric reduces locally to $\omega_\alpha := \overline{\partial} \alpha + \partial \overline{\alpha}$ for some $\alpha \in \Lambda^{1,0}$. Based on this observation, in [38] the author and Tian reduced the pluriclosed flow to a degenerate parabolic equation for a $(1,0)$-form. Locally this equation takes the form

$$\frac{\partial}{\partial t} \alpha = \overline{\partial}^*_{g_\alpha} \omega_\alpha - \frac{\sqrt{-1}}{2} \partial \log \det g_\alpha. \tag{1.2}$$

Making this reduction global requires certain choices of background data, made precise in §3. This is akin to the reduction of Kähler-Ricci flow to the parabolic complex Monge Ampere equation, which requires certain choices of background data. Equation (1.2) is a degenerate parabolic equation for α, which can be further reduced to the parabolic complex Monge-Ampere in the case that the underlying metric is Kähler. In this paper we approach the pluriclosed flow entirely from the point of view of equation (1.2), which as it turns out holds the key to many useful differential inequalities which can be used to obtain a priori estimates.

The first principal estimate is an a priori C^α estimate for the metric in the presence of upper and lower bounds on the metric (Theorem 1.7 below). Thinking of the pluriclosed flow as a parabolic system of equations for the Hermitian metric g, this estimate is analogous to the DeGiorgi-Nash-Moser/Krylov-Safonov [10, 30, 29, 25, 26] estimate for uniformly parabolic equations. On the other hand, the corresponding estimate for the Kähler-Ricci flow is analogous to a $C^{2,\alpha}$ estimate for the potential, and the techniques of Evans-Krylov [14, 24] can be applied to obtain this estimate. We emphasize that these are only analogies, as indeed we do NOT have a scalar reduction for our system, let alone a convex one, so the result of Evans-Krylov cannot apply. Moreover, the DeGiorgi-Nash-Moser/Krylov-Safonov results are known to be false in general for *systems* of equations [11], which is the setting here. Lastly, we point out that in complex coordinates the pluriclosed flow is a quasilinear parabolic system with "first order quadratic nonlinearity." This is the type of nonlinearity arising for instance in harmonic maps, and Struwe [41] has shown the corresponding C^α estimate in this setting with the further assumption that the nonlinear term is small with respect to ellipticity constants, an assumption not available in this setting.

Despite the complexity of the pluriclosed flow system, our method of proof is related to that of Evans-Krylov for the scalar PDE setting, which we briefly recount here. Recall that these results yield $C^{2,\alpha}$ estimates for $C^{1,1}$ solutions to uniformly parabolic, fully nonlinear, *convex* equations. This convexity is exploited most crucially to show that every second directional derivative of the given function is a subsolution to a uniformly parabolic equation. One thus obtains a weak Harnack estimate which is used in conjunction with the original fully nonlinear equation to obtain the full $C^{2,\alpha}$ regularity. This method certainly does not directly apply since we not even have a scalar PDE underlying the pluriclosed flow, let alone a convex one.

Nonetheless, by a careful study of the 1-form potential α along the pluriclosed flow, we discover a judicious combination of the first derivatives of α into a Hermitian $2n \times 2n$ matrix W such that $W(v,v)$ is a subsolution to a uniformly parabolic equation for every v, and such that $\det W = 1$. The form of this matrix was inspired by the author's previous joint work with M. Warren [40] in which a closely related quantity was discovered and applied to obtain Evans-Krylov type regularity for certain nonconvex parabolic equations, arising partly from the pluriclosed flow in the generalized Kähler setting [35], where the flow reduces to a scalar PDE. In that case the matrix playing the role of W admits a clear interpretation as the Hessian of a function obtained by applying a partial Legendre transformation to the solution to the given PDE. Somewhat miraculously, a very similar quantity obeys remarkable partial differential inequalities for the general pluriclosed flow, which is a parabolic system. This matrix W admits an interpretation as a "Born-Infeld" metric on the generalized tangent bundle $T \oplus T^*$. This seems at first glance to deepen the apparent connection between the pluriclosed flow and generalized geometry [34, 35, 39]. We will discuss this further in §5.

Before stating the theorem we introduce a piece of notation. For a Hermitian manifold (M^{2n}, J, g), let h denote an auxiliary Hermitian metric, and let $\Upsilon(g,h) = \nabla_g - \nabla_h$ be the difference of the two Chern connections. Furthermore, let

$$f_k = f_k(g,h) := \sum_{j=0}^{k} \left|\nabla_g^j \Upsilon(g,h)\right|^{\frac{2}{1+j}}.$$

The quantity f_k is a natural measure of the $k+2$-th derivatives of the metric which scales as the inverse of the metric.

Theorem 1.7. *Let (M^{2n}, J) be a compact complex manifold. Suppose g_t is a solution to the pluriclosed flow on $[0,\tau)$, $\tau \leq 1$, with α_t a solution to the $(\hat{g}_t, h, \mu)$-reduced flow (see §3.1). Suppose there exist constants λ, Λ such that*

$$\lambda g_0 \leq g_t \leq \Lambda g_0, \qquad \left|\partial\alpha\right|^2 \leq \Lambda. \tag{1.3}$$

Given $k \in \mathbb{N}$ there exists a constant $C = C(n,k,g_0,\hat{g},h,\mu,\lambda,\Lambda)$ such that

$$\sup_{M \times \{t\}} t f_k(g_t, h) \leq C.$$

The second principal estimate of this paper is a general upper bound for the metric in terms of a lower bound. The proof exploits the very favorable evolution equations arising from the 1-form reduction of pluriclosed flow to control certain torsion terms arising in the evolution of metric quantities.

Theorem 1.8. *Let (M^{2n}, J) be a compact complex manifold. Suppose g_t is a solution to the pluriclosed flow on $[0,\tau)$, with α_t a solution to the $(\hat{g}_t, h, \mu)$-reduced flow. Assume there is a constant λ such that for all $t \in [0,\tau)$,*

$$\lambda g_0 \leq g_t.$$

There exists a constant $\Lambda = \Lambda(n, g_0, \hat{g}, h, \mu, \lambda)$ such that for all $t \in [0,\tau)$,

$$g_t \leq \Lambda(1+t) g_0, \qquad \left|\partial\alpha\right|^2 \leq \Lambda.$$

Here is an outline of the rest of this paper. In §2 we recall notation and some basic facts concerning the pluriclosed flow. Then in §3 we develop the one-form reduction of the pluriclosed flow introduced in [37]. Next in §4 we introduce the "torsion potential" along a solution to pluriclosed flow. In §5 we give the proof of Theorems 1.7 and 1.8. We use these in §6 and §7 to prove Theorems 1.1, 1.2 and 1.3, and Corollary 1.4.

Acknowledgements: The author would like to thank Marco Gualtieri and Luis Ugarte for useful conversations.

2. Background

2.1. **Different Faces of Pluriclosed Flow.** In this subsection we give three equivalent formulations of the pluriclosed flow equation, each of which is useful in easily displaying certain properties of the equation.

2.1.1. *Hodge operators formulation.* First we express the pluriclosed flow equation using differential operators appearing in Hodge theory. This point of view makes manifest the fact that the flow preserves the pluriclosed condition, and moreover is essential to constructing the 1-form formulation of pluriclosed flow analyzed below. Let (M^{2n}, g, J) be a Hermitian manifold. One has the natural decomposition $d = \partial + \overline{\partial}$, and $d^* = \partial^* + \overline{\partial}^*$. Recall that ω is pluriclosed if $\partial\overline{\partial}\omega = 0$. Since the local generality of pluriclosed metrics is that of a $(1,0)$-form (see Lemma 2.1), to find a parabolic flow of such metrics it is natural to consider the ansatz $\frac{\partial}{\partial t}\omega = \partial\alpha + \overline{\partial\alpha} + \gamma$, where $\alpha \in \Lambda^{0,1}$ and γ is closed. Taking inspiration from Kähler-Ricci flow, it is natural to let $\gamma = -c_1(M, \omega)$. Since we want a second-order flow, one needs α to be a first-order operator on the metric, and one has little choice but to set $\alpha = \partial^*\omega$. This point of view leads one to the pluriclosed flow equation

$$\frac{\partial}{\partial t}\omega = \partial\partial_g^*\omega + \overline{\partial\partial_g^*}\omega + \sqrt{-1}\partial\overline{\partial}\log\det g.$$

As shown in [36], this is a strictly parabolic equation with pluriclosed initial condition ω_0, and admits short-time solutions on compact manifolds.

2.1.2. *Chern connection formulation.* Given (M^{2n}, g, J) a Hermitian manifold, the Chern connection is the unique connection ∇ on $T^{1,0}(M)$ such that $\nabla g \equiv 0$, $\nabla J \equiv 0$ and the torsion of ∇ has vanishing $(1,1)$ piece. In local complex coordinates the connection coefficients are

$$\Gamma_{ij}^k = g^{\bar{l}k} g_{j\bar{l},i}.$$

The torsion of the Chern connection takes the form

$$T_{ij\overline{k}} = g_{l\overline{k}}\left[\Gamma_{ij}^l - \Gamma_{ki}^l\right] = g_{j\overline{k},i} - g_{i\overline{k},j}.$$

The metric is Kähler if and only if $T \equiv 0$. Also the Chern curvature takes the form

$$\Omega_{i\bar{j}k}^l = -\partial_{\bar{j}}\Gamma_{ik}^l = -\partial_{\bar{j}}\left(g^{\overline{m}l}g_{k\overline{m},i}\right) = -g^{\overline{m}l}g_{k\overline{m},i\bar{j}} + g^{\overline{m}p}g^{\overline{q}l}g_{p\overline{q},\bar{j}}g_{k\overline{m},i}.$$

Due to the fact that ∇, in general, has torsion, there are various "Ricci curvatures" which can be defined using this connection. We concentrate on one of them,

$$S_{i\bar{j}} = g^{\overline{q}p}\Omega_{p\overline{q}i\bar{j}}.$$

Observe that this is the "Ricci curvature" which appears in the theory of Hermitian Yang-Mills theory, although in that setting the connection ∇ is some Hermitian connection on a complex vector bundle over M, which is independent of the Hermitian metric chosen on M. We also define a certain quadratic expression in torsion, namely

$$Q_{i\bar{j}} = g^{\bar{l}k}g^{\overline{n}m}T_{ik\overline{n}}T_{\bar{j}\bar{l}m}.$$

With these definitions made, we can express the pluriclosed flow equation ([36] Proposition 3.3) as

$$\frac{\partial}{\partial t}g = -S + Q. \tag{2.1}$$

2.1.3. *Bismut connection formulation.* Let (M^{2n}, g, J) be a Hermitian manifold. Recall the operator $d^c = \sqrt{-1}(\overline{\partial} - \partial)$, and note that in particular one has

$$d^c\omega(X, Y, Z) = -d\omega(JX, JY, JZ).$$

The Bismut connection [5] is the unique connection on TM for which $\nabla g \equiv 0$, $\nabla J \equiv 0$, and which has skew symmetric torsion. It follows that

$$g(\nabla^B_X Y, Z) = g(\nabla^{LC}_X Y, Z) + \frac{1}{2} d^c\omega(X, Y, Z).$$

This connection induces a connection on the canonical bundle, and its curvature can be computed as

$$\rho^B(X, Y) = g(R^B(X, Y)e_i, Je_i).$$

The form ρ is closed by the Bianchi identity. But since J is no longer parallel, ρ is not necessarily a $(1,1)$-form. A computation shows that then the pluriclosed flow is equivalent to

$$\frac{\partial}{\partial t}\omega = -\rho_B^{1,1}.$$

Using this formulation one is able to show that solutions to the pluriclosed flow are gauge-modified solutions to the B-field renormalization group flow, which then exhibits pluriclosed flow as a gradient flow which admits Perelman-type monotonicity formulas [37].

Lastly, for certain applications it is natural to add a normalizing term to the pluriclosed flow similar to the normalization frequently imposed on Kähler-Ricci flow. Consider

$$\frac{\partial}{\partial t}\omega = -(\rho^B)^{1,1} - \omega \tag{2.2}$$

2.2. **Local generality of pluriclosed metrics and Aeppli classes.** We recall that, locally, any Kähler metric may be expressed as $\sqrt{-1}\partial\overline{\partial}f$ for some smooth real function f. In this section we prove a similar result for pluriclosed metrics, exhibiting the local generality of such metrics in terms of $(1,0)$-forms.

Lemma 2.1. *Let $U \subset \mathbb{C}^n$ be an open subset homeomorphic to a ball, and suppose $\omega \in \Lambda^{1,1}_{\mathbb{R}}$ is a pluriclosed form on U. There exists $\alpha \in \Lambda^{1,0}$ such that*

$$\omega = \overline{\partial}\alpha + \partial\overline{\alpha}.$$

Proof. Since the form $\partial\omega$ is d-closed, so by the local $\partial\overline{\partial}$ lemma we obtain $\beta \in \Lambda^{1,0}$ such that

$$\partial\omega = \partial\overline{\partial}\beta.$$

Now consider the form

$$\gamma := \omega - \overline{\partial}\beta - \partial\overline{\beta}.$$

Note that

$$\overline{\partial}\gamma = \overline{\partial}\omega - \overline{\partial}\partial\overline{\beta} = 0, \qquad \partial\gamma = \partial\omega - \partial\overline{\partial}\beta = 0.$$

Since $\gamma \in \Lambda^{1,1}_{\mathbb{R}}$ is d-closed, it follows again by the $\partial\overline{\partial}$-lemma that there exists $f \in C^\infty(M, \mathbb{R})$ such that $\gamma = \sqrt{-1}\partial\overline{\partial}f$. Finally, set

$$\alpha = \beta - \frac{\sqrt{-1}}{2}\partial f.$$

We then directly compute

$$\overline{\partial}\alpha + \partial\overline{\alpha} = \overline{\partial}\beta + \partial\overline{\beta} + \sqrt{-1}\partial\overline{\partial}f = \overline{\partial}\beta + \partial\overline{\beta} + \gamma = \omega.$$

□

Given this lemma, on compact manifolds it is natural to consider pluriclosed metrics up to equivalence of adding $\overline{\partial}\alpha + \partial\overline{\alpha}$, in analogy with Kähler classes in Kähler geometry.

Definition 2.2. Let (M^{2n}, ω, J) be a complex manifold with pluriclosed metric ω. We set

$$\mathcal{H}_\omega := \{\alpha \in \Lambda^{1,0} |\ \omega_\alpha := \omega + \overline{\partial}\alpha + \partial\overline{\alpha} > 0\}.$$

Note also that each metric $\omega_\alpha \in \mathcal{H}_\omega$ can be described by an infinite dimensional equivalence class, by adding any element of the image of $\partial : C^\infty(M) \to \Lambda^{1,0}$. We formalize this in the next definition.

Definition 2.3. Let (M^{2n}, J) be a complex manifold. We let

$$\mathcal{G} = \{\partial f | f \in C^\infty(M, \mathbb{R})\}.$$

Moreover, we let $\Lambda^{1,0}/\mathcal{G}$ denote the corresponding space of equivalence classes of $(0,1)$-forms.

Remark 2.4. Thus, as the Kähler form is written as

$$\omega^\alpha = \omega + \overline{\partial}\alpha + \partial\overline{\alpha},$$

we have the coordinate expression

$$\omega^\alpha_{i\overline{j}} = \omega_{i\overline{j}} + \left(\overline{\partial}\alpha + \partial\overline{\alpha}\right)_{i\overline{j}} = \omega_{i\overline{j}} + \left(\alpha_{\overline{j},i} - \alpha_{i,\overline{j}}\right).$$

This means that the *metric coefficients* take the form

$$g^\alpha_{i\overline{j}} = g_{i\overline{j}} - \sqrt{-1}\left(\alpha_{\overline{j},i} - \alpha_{i,\overline{j}}\right) = g_{i\overline{j}} + \sqrt{-1}\left(\alpha_{i,\overline{j}} - \alpha_{\overline{j},i}\right). \tag{2.3}$$

2.3. **The positive cone.**

Definition 2.5. Let (M^{2n}, J) be a complex manifold. Let

$$H^{1,1}_{\partial+\overline{\partial}} := \frac{\left\{\psi \in \Lambda^{1,1}_{\mathbb{R}} |\ \partial\overline{\partial}\psi = 0\right\}}{\left\{\overline{\partial}\alpha + \partial\overline{\alpha} |\ \alpha \in \Lambda^{1,0}\right\}}.$$

This is referred to as the $(1,1)$-Aeppli cohomology, defined in [1]. Next, in analogy with the Kähler cone, we next define the cone of classes in $H^{1,1}_{\partial+\overline{\partial}}$ which admit pluriclosed metrics.

Definition 2.6. Let (M^{2n}, J) be a complex manifold. Let

$$\mathcal{P} := \left\{[\psi] \in H^{1,1}_{\partial+\overline{\partial}} \mid \exists\ \omega \in [\psi], \omega > 0\right\}.$$

The space $\mathcal{P}$ is an open cone in $\mathcal{H}^{1,1}_{\partial+\overline{\partial}}$, which is nonempty if and only if M admits pluriclosed metrics. We refer to $\mathcal{P}$ as the *positive cone*.

2.4. **The formal existence time.** Observe that, if ω_t is a solution to pluriclosed flow, then $[\omega_t]$ defines a path in $\mathcal{P}$. Moreover, since $H^{1,1}_{\mathbb{R}} \subset H^{1,1}_{\partial+\overline{\partial}}$, we can interpret the first Chern class of (M, J) as an element of $H^{1,1}_{\partial+\overline{\partial}}$. With this point of view we observe that a solution to pluriclosed flow satisfies

$$[\omega_t] = [\omega_0] - tc_1.$$

Meanwhile, a solution to (2.2) satisfies

$$[\omega_t] = [-\rho(h) + e^{-t}(\omega_0 + \rho(h))]$$

Certainly the flow cannot exist smoothly if the boundary of $\mathcal{P}$ is reached along these paths. We state this for emphasis.

Definition 2.7. Let (M^{2n}, J) be a compact complex manifold, and suppose g_0 is a pluriclosed metric on M. Let

$$\tau^*(g_0) := \sup\{t | [\omega_0] - tc_1 \in \mathcal{P}\}.$$

Lemma 2.8. *Let (M^{2n}, J) be a compact complex manifold, and suppose g_0 is a pluriclosed metric on M. If τ denotes the maximal existence time of the solution to pluriclosed flow with initial condition g_0, then $\tau \leq \tau^*(g_0)$.*

The main guiding conjecture behind our study of pluriclosed flow is that in fact reaching the boundary of the cone is the *only* way to have a singularity.

Conjecture 2.9. Weak existence conjecture: *Let (M^{2n}, g_0, J) be a compact complex manifold with pluriclosed metric. Then the solution to pluriclosed flow with initial condition g_0 exists smoothly on $[0, \tau^*(g_0))$.*

3. The 1-form reduction of pluriclosed flow

In this section we develop a reduction of the pluriclosed flow into a degenerate parabolic system for a $(1,0)$-form. Since a Kähler metric depends locally on a single function and the Ricci flow preserves the Kähler condition, it is natural to expect that the flow reduces to that of a single function, up to finite dimensional (i.e. cohomological) obstructions. This is easily borne out by computations to show that the Kähler-Ricci flow is locally equivalent to the parabolic complex Monge-Ampere equation.

Applying a similar line of thought, based on Lemma 2.1 one would expect the pluriclosed flow to reduce to a flow of a $(1,0)$-form. Note however that, on a compact Kähler manifold, the kernel of $\sqrt{-1}\partial\overline{\partial}$ acting on functions is given just by constant functions, and therefore one expects a strictly parabolic equation for the potential function, as indeed is the case. For the pluriclosed case, there is always an infinite dimensional kernel to the description of a pluriclosed metric by a $(1,0)$-form (see Definitions 2.2, 2.3). Thus one expects to be able to reduce the pluriclosed flow to a degenerate parabolic equation for a $(1,0)$-form, and this was indeed observed in ([37] Theorem 5.16, Proposition 5.18) (n.b. our notation and conventions within differ slightly from that paper).

Our purpose in this section is to further develop this point of view on the pluriclosed flow to obtain a priori estimates. At first the value of this reduction to a $(1,0)$-form may seem doubtful since we have taken an equation which is *strictly* parabolic in complex coordinates and reduced to one which is *degenerate* parabolic. Nonetheless this reduction gives us access to quantities which are otherwise invisible, including a gauge-invariant potential for the torsion of the flowing metric which obeys a remarkably simple evolution equation along the flow, as detailed in §4.

3.1. Reduction of pluriclosed flow.

Let (M^{2n}, g_t, J) be a smooth solution to pluriclosed flow on $[0,\tau]$. We fix a background Hermitian metric h. By Lemma 2.8, we know $\tau < \tau^*(g_0)$, and hence there exists $\mu \in \Lambda^{1,0}$ such that

$$\hat{\omega}_\tau := \omega_0 - \tau\rho(h) + \overline{\partial}\mu + \partial\overline{\mu} > 0. \tag{3.1}$$

Now consider the smooth one-parameter family of Kähler forms

$$\hat{\omega}_t := \frac{t}{\tau}\hat{\omega}_\tau + \frac{(\tau - t)}{\tau}\omega_0.$$

Observe that, as Aeppli cohomology classes,

$$\frac{\partial}{\partial t}[\hat{\omega}_t] = \frac{1}{\tau}\left[\hat{\omega}_\tau - \omega_0\right] = \left[\sqrt{-1}\partial\overline{\partial}\log\det h + \frac{1}{\tau}\left(\overline{\partial}\mu + \partial\overline{\mu}\right)\right] = -c_1$$

and hence $[\hat{\omega}_t] = [\omega_0] - tc_1$, and so ω_t serves as an appropriate family of background metrics.

Definition 3.1. Let (M^{2n}, g_t, J) be a smooth solution to pluriclosed flow on $[0,\tau]$. Given choices $\hat{g}_t, h, \mu$ as above, for a one parameter family $\alpha_t \in \Lambda^{1,0}$ let

$$\omega_\alpha := \hat{\omega}_t + \overline{\partial}\alpha_t + \partial\overline{\alpha_t}.$$

We say that a one-parameter family $\alpha_t \in \Lambda^{1,0}$ is a solution to $(\hat{g}_t, h, \mu)$-*reduced pluriclosed flow* if

$$\begin{aligned} \frac{\partial}{\partial t}\alpha &= \overline{\partial}^*_{g_\alpha}\omega_\alpha - \frac{\sqrt{-1}}{2}\partial \log \frac{\det g_\alpha}{\det h} - \frac{\mu}{\tau}, \\ \alpha_0 &= 0. \end{aligned} \tag{3.2}$$

In the sequel, if no further context is given we will refer to a solution of 1-form reduced pluriclosed flow with some arbitrary choices of $\hat{g}_t, h$ and μ having been made. Observe that there is always an infinite dimensional equivalence class of solutions corresponding to the $\mathcal{G}$-orbit.

Lemma 3.2. *Let (M^{2n}, g_t, J) be a smooth solution to pluriclosed flow on $[0,\tau]$. Given choices $\hat{g}_t, h, \mu$ as above, there exists a solution α_t to $(\hat{g}_t, h, \mu)$-reduced pluriclosed flow, and for any solution α_t to (3.2), one has $g_{\alpha_t} = g_t$.*

Proof. Given the setup, we let α_t be the solution to the ordinary differential equation

$$\begin{aligned} \frac{\partial}{\partial t}\alpha &= \overline{\partial}^*_{g_t}\omega_t - \frac{\sqrt{-1}}{2}\partial \log \frac{\det g_t}{\det h} - \frac{\mu}{\tau} \\ \alpha_0 &= 0. \end{aligned} \tag{3.3}$$

We thus compute

$$\begin{aligned} \frac{\partial}{\partial t}\omega_{\alpha_t} &= \frac{\partial}{\partial t}\hat{\omega}_t + \overline{\partial}\dot{\alpha} + \partial\dot{\overline{\alpha}} \\ &= \left(\sqrt{-1}\partial\overline{\partial}\log\det h + \frac{1}{\tau}(\overline{\partial}\mu + \partial\overline{\mu})\right) \\ &\quad + \overline{\partial}\left(\overline{\partial}^*_{g_t}\omega_t - \frac{\sqrt{-1}}{2}\partial\log\frac{\det g_t}{\det h} - \frac{\mu}{\tau}\right) + \partial\left(\partial^*_{g_t}\omega_t + \frac{\sqrt{-1}}{2}\overline{\partial}\log\frac{\det g_t}{\det h} - \frac{\overline{\mu}}{\tau}\right) \\ &= \partial\partial^*_{g_t}\omega_t + \overline{\partial}\overline{\partial}^*_{g_t}\omega_t + \sqrt{-1}\partial\overline{\partial}\log\det g. \end{aligned}$$

It follows that $\frac{\partial}{\partial t}(\omega_t - \omega_{\alpha_t}) = 0$, and so $\omega_t = \omega_{\alpha_t}$. Plugging this into (3.3) gives that α_t is indeed a solution to (3.2). Also, given *any* solution to (3.2), a calculation nearly identical to that above yields that the family of metrics $g_{\alpha_t} = g_t$. □

Remark 3.3. Note that this lemma does not claim an a priori construction of the solution to (3.2), in the sense that we already require the solution to pluriclosed flow to obtain the reduction. Nonetheless it is possible to construct an a priori solution, and we comment on this below.

3.2. **The one-form differential operator.** In this subsection we begin our analysis of 1-form reduced pluriclosed flow. We simplify matters by defining a differential operator lying at the heart of the analysis.

Definition 3.4. Let (M^{2n}, J) be a complex manifold. Let $\hat{g}$ denote a pluriclosed metric on M, let h denote any Hermitian metric on M , and $\mu \in \Lambda^{1,0}$. Given $\alpha \in \mathcal{H}_{\hat{\omega}}$, let

$$\Psi(\hat{g}, h, \mu, \alpha) = \overline{\partial}^*_{g_\alpha}\omega_\alpha - \frac{\sqrt{-1}}{2}\partial\log\frac{\det g_\alpha}{\det h} + \mu.$$

Since the main object of interest in this operator is α, we will frequently abbreviate $\Psi(\alpha) = \Psi(\hat{g}, h, \mu, \alpha)$ when background choices of the other objects is clear from context.

Lemma 3.5. *Given $\hat{g}, h, \mu, \alpha$ as above, in local complex coordinates we have*

$$\Psi(\hat{g}, h, \mu, \alpha)_i = g_\alpha^{\overline{q}p}\left[\alpha_{i,p\overline{q}} - \frac{1}{2}\left(\alpha_{p,\overline{q}i} + \alpha_{\overline{q},pi}\right)\right] + \sqrt{-1} g_\alpha^{\overline{q}p}\left[\frac{1}{2}\hat{g}_{p\overline{q},i} - \hat{g}_{i\overline{q},p}\right] + \frac{\sqrt{-1}}{2} h^{\overline{q}p} h_{p\overline{q},i} + \mu_i.$$

Proof. A basic calculation in coordinates shows that

$$\begin{aligned}\left(\overline{\partial}_g^* \omega - \frac{\sqrt{-1}}{2}\partial \log \frac{\det g}{\det h}\right)_i =&\ \sqrt{-1} g^{\overline{q}p}\left[g_{p\overline{q},i} - g_{i\overline{q},p}\right] - \frac{\sqrt{-1}}{2}\left[g^{\overline{q}p} g_{p\overline{q},i} - h^{\overline{q}p} h_{p\overline{q},i}\right]\\ =&\ \sqrt{-1} g^{\overline{q}p}\left[\frac{1}{2} g_{p\overline{q},i} - g_{i\overline{q},p}\right] + \frac{\sqrt{-1}}{2} h^{\overline{q}p} h_{p\overline{q},i}.\end{aligned}$$

Using this we have

$$\begin{aligned}\Psi(\hat{g}, h, 0, \alpha)_i =&\ \left(\overline{\partial}_{g_\alpha}^* \omega_\alpha - \frac{\sqrt{-1}}{2}\partial \log \frac{\det g_\alpha}{\det h}\right)_i\\ =&\ \sqrt{-1} g_\alpha^{\overline{q}p}\left[\frac{1}{2} g^\alpha_{p\overline{q},i} - g^\alpha_{i\overline{q},p}\right] + \frac{\sqrt{-1}}{2} h^{\overline{q}p} h_{p\overline{q},i}\\ =&\ g_\alpha^{\overline{q}p}\left[\alpha_{i,p\overline{q}} - \frac{1}{2}\left(\alpha_{p,\overline{q}i} + \alpha_{\overline{q},pi}\right)\right] + \sqrt{-1} g_\alpha^{\overline{q}p}\left[\frac{1}{2}\hat{g}_{p\overline{q},i} - \hat{g}_{i\overline{q},p}\right] + \frac{\sqrt{-1}}{2} h^{\overline{q}p} h_{p\overline{q},i}.\end{aligned}$$

The result follows. □

Lemma 3.6. *Given $\hat{g}, h, \mu, \alpha$ as above, in local complex coordinates we have*

$$\Psi(\hat{g}, h, \mu, \alpha)_i = \Delta_{g_\alpha}\alpha_i - \left(T_\alpha \circ \overline{\partial}\alpha\right)_i + \sqrt{-1} g_\alpha^{\overline{q}p} \hat{T}_{ip\overline{q}} - \frac{\sqrt{-1}}{2}\nabla_i\left[\operatorname{tr}_{g_\alpha}\hat{g} + \log\frac{\det g_\alpha}{\det h} - 2\sqrt{-1}\Re\overline{\nabla}^*\alpha\right] + \mu_i,$$

where

$$\left(T_\alpha \circ \overline{\partial}\alpha\right)_i = g_\alpha^{\overline{l}k} g_\alpha^{\overline{q}p} T^\alpha_{ik\overline{q}} \nabla_{\overline{l}}\alpha_p.$$

Proof. We begin with three preliminary calculations,

$$\begin{aligned}\Delta_{g_\alpha}\alpha_i =&\ g_\alpha^{\overline{l}k}\left[\nabla\overline{\nabla}\alpha\right]_{k\overline{l}i}\\ =&\ g_\alpha^{\overline{l}k}\left[\partial_k\left[\overline{\nabla}\alpha\right]_{\overline{l}i} - \Gamma^p_{ki}(\overline{\nabla}\alpha)_{\overline{l}p}\right]\\ =&\ g_\alpha^{\overline{l}k}\left[\alpha_{i,k\overline{l}} - g_\alpha^{\overline{q}p} g^\alpha_{i\overline{q},k}\nabla_{\overline{l}}\alpha_p\right].\end{aligned}$$

Also

$$\begin{aligned}\left(\nabla\nabla^*\alpha\right)_i =&\ g_\alpha^{\overline{l}k}\nabla_i\nabla_{\overline{l}}\alpha_k\\ =&\ g_\alpha^{\overline{l}k}\left[\partial_i(\nabla_{\overline{l}}\alpha_k) - \Gamma^p_{ik}\nabla_{\overline{l}}\alpha_p\right]\\ =&\ g_\alpha^{\overline{l}k}\left[\alpha_{k,\overline{l}i} - g_\alpha^{\overline{q}p} g^\alpha_{k\overline{q},i}\nabla_{\overline{l}}\alpha_p\right].\end{aligned}$$

Next

$$\begin{aligned}\left(\nabla\overline{\nabla}^*\overline{\alpha}\right)_i =&\ g_\alpha^{\overline{l}k}\nabla_i\nabla_k\alpha_{\overline{l}}\\ =&\ g_\alpha^{\overline{l}k}\left[\partial_i(\nabla_k\alpha_{\overline{l}}) - \Gamma^p_{ik}(\nabla_p\alpha_{\overline{l}})\right]\\ =&\ g_\alpha^{\overline{l}k}\left[\alpha_{\overline{l},ik} - g_\alpha^{\overline{q}p} g^\alpha_{k\overline{q},i}\nabla_p\alpha_{\overline{l}}\right].\end{aligned}$$

Combining these yields

$$\begin{aligned}
& g_\alpha^{\bar{l}k}\left[\alpha_{i,\bar{l}k}-\frac{1}{2}\left(\alpha_{k,\bar{l}i}+\alpha_{\bar{l},ki}\right)\right] \\
&= \Delta_{g_\alpha}\alpha_i-\frac{1}{2}\nabla_i\left(\nabla^*\alpha+\overline{\nabla}^*\overline{\alpha}\right)+g_\alpha^{\bar{l}k}g_\alpha^{\overline{q}p}\left[g^\alpha_{i\overline{q},k}\nabla_{\bar{l}}\alpha_p-\frac{1}{2}g^\alpha_{k\overline{q},i}\nabla_{\bar{l}}\alpha_p-\frac{1}{2}g^\alpha_{k\overline{q},i}\nabla_p\alpha_{\bar{l}}\right] \\
&= \Delta_{g_\alpha}\alpha_i-\frac{1}{2}\nabla_i\left(\nabla^*\alpha+\overline{\nabla}^*\overline{\alpha}\right)+g_\alpha^{\bar{l}k}g_\alpha^{\overline{q}p}\left[T^\alpha_{ki\overline{q}}\nabla_{\bar{l}}\alpha_p+\frac{1}{2}g^\alpha_{k\overline{q},i}(\nabla_{\bar{l}}\alpha_p-\nabla_p\alpha_{\bar{l}})\right] \\
&= \Delta_{g_\alpha}\alpha_i-\frac{1}{2}\nabla_i\left(\nabla^*\alpha+\overline{\nabla}^*\overline{\alpha}\right)+g_\alpha^{\bar{l}k}g_\alpha^{\overline{q}p}\left[T^\alpha_{ki\overline{q}}\nabla_{\bar{l}}\alpha_p+\frac{\sqrt{-1}}{2}g^\alpha_{k\overline{q},i}(\hat{g}_{p\bar{l}}-g^\alpha_{p\bar{l}})\right] \\
&= \Delta_{g_\alpha}\alpha_i-\frac{1}{2}\nabla_i\left(\nabla^*\alpha+\overline{\nabla}^*\overline{\alpha}\right)+g_\alpha^{\bar{l}k}g_\alpha^{\overline{q}p}T^\alpha_{ki\overline{q}}\nabla_{\bar{l}}\alpha_p-\frac{\sqrt{-1}}{2}g_\alpha^{\bar{l}k}g^\alpha_{k\bar{l},i}+\frac{\sqrt{-1}}{2}g_\alpha^{\bar{l}k}g_\alpha^{\overline{q}p}\hat{g}_{p\bar{l}}g^\alpha_{k\overline{q},i}.
\end{aligned}$$

Plugging this into the result of Lemma 3.5 yields

$$\begin{aligned}
\Psi_i = {} & \Delta_{g_\alpha}\alpha_i-\nabla_i\Re\nabla^*\alpha+g_\alpha^{\bar{l}k}g_\alpha^{\overline{q}p}T^\alpha_{ki\overline{q}}\nabla_{\bar{l}}\alpha_p-\frac{\sqrt{-1}}{2}g_\alpha^{\bar{l}k}g^\alpha_{k\bar{l},i}+\frac{\sqrt{-1}}{2}g_\alpha^{\bar{l}k}g_\alpha^{\overline{q}p}\hat{g}_{p\bar{l}}g^\alpha_{k\overline{q},i} \\
& +\sqrt{-1}g_\alpha^{\overline{q}p}\left[\frac{1}{2}\hat{g}_{p\overline{q},i}-\hat{g}_{i\overline{q},p}\right]+\frac{\sqrt{-1}}{2}h^{\overline{q}p}h_{p\overline{q},i}+\mu_i.
\end{aligned}$$

One further simplification yields

$$\begin{aligned}
\sqrt{-1}g_\alpha^{\overline{q}p}\left[-\hat{g}_{i\overline{q},p}+\frac{1}{2}\hat{g}_{p\overline{q},i}+\frac{1}{2}\hat{g}_\alpha^{\bar{l}k}g_{p\bar{l}}g^\alpha_{k\overline{q},i}\right] &= \sqrt{-1}g_\alpha^{\overline{q}p}\left[\hat{T}_{ip\overline{q}}-\frac{1}{2}\hat{g}_{p\overline{q},i}+\frac{1}{2}g_\alpha^{\bar{l}k}\hat{g}_{p\bar{l}}g^\alpha_{k\overline{q},i}\right] \\
&= \sqrt{-1}g_\alpha^{\overline{q}p}\hat{T}_{ip\overline{q}}-\frac{\sqrt{-1}}{2}\nabla_i\operatorname{tr}_{g_\alpha}\hat{g}.
\end{aligned}$$

Combining these calculations yields the final result. ☐

Proposition 3.7. *Given $\hat{g},h,\mu$ as above, the map $\Psi(\hat{g},h,\mu,\cdot):\Lambda^{1,0}\to\Lambda^{1,0}$ is a second order degenerate elliptic operator.*

Proof. A simple calculation using Lemma 3.5 shows that

$$\left[\sigma\left(D_\alpha\Psi\right)(\beta)\right](\xi)_i = g_\alpha^{\overline{q}p}\left[\beta_i\xi_p\xi_{\overline{q}}-\frac{1}{2}\left(\beta_p\xi_{\overline{q}}\xi_i+\beta_{\overline{q}}\xi_p\xi_i\right)\right].$$

Thus

$$\begin{aligned}
\left\langle\left[\sigma\left(D_\alpha\Psi\right)(\beta)\right](\xi),\beta\right\rangle &= \left|\beta\right|^2_{g_\alpha}\left|\xi\right|^2-\frac{1}{2}g_\alpha^{\bar{j}i}g_\alpha^{\overline{q}p}\left(\overline{\beta}_{\bar{j}}\beta_p\xi_{\overline{q}}\xi_i+\beta_{\overline{q}}\xi_p\xi_i\right) \\
&= \left|\beta\right|^2_{g_\alpha}\left|\xi\right|^2-\left|\left\langle\beta,\xi\right\rangle_{g_\alpha}\right|^2 \\
&\geq 0,
\end{aligned}$$

where the last line follows by the Cauchy-Schwarz inequality. ☐

Remark 3.8. By direct calculation one can show that the kernel corresponds to the image of $\partial:C^\infty(M)\to\Lambda^{1,0}$, corresponding to the $\mathcal{G}$-orbit, as expected.

3.3. **A splitting of the 1-form system.** In this subsection we exhibit an essentially canonical way to "split" the 1-form equation into a simpler 1-form equation and an equation for a scalar quantity.

Proposition 3.9. *Let (M^{2n}, g_t, J) be a solution to pluriclosed flow. Given choices $\hat{g}_t, h, \mu$ as above, suppose (β_t, f_t) is a one-parameter family of $(1,0)$-forms β and smooth functions f such that*

$$\begin{aligned}
\frac{\partial}{\partial t}\beta =&\ \Delta_{g_t}\beta - T_{g_t}\circ \overline{\partial}\beta + \sqrt{-1}\operatorname{tr}_{g_\alpha} T + \mu,\\
\frac{\partial}{\partial t} f =&\ \Delta_{g_t} f + \operatorname{tr}_{g_t} g + \log \frac{\det g_t}{\det h},\\
\alpha_0 =&\ \beta_0 - \sqrt{-1}\partial f_0.
\end{aligned} \tag{3.4}$$

Then $\alpha_t := \beta_t - \sqrt{-1}\partial f_t$ is a solution to the 1-form flow.

Proof. We directly compute that

$$\begin{aligned}
\frac{\partial}{\partial t}\alpha =&\ \frac{\partial}{\partial t}\beta - \sqrt{-1}\partial\frac{\partial}{\partial t} f\\
=&\ \Delta_{g_t}\beta - T_\alpha\circ\overline{\partial}\beta + \sqrt{-1}\operatorname{tr}_{g_\alpha} T + \mu - \sqrt{-1}\partial\left(\Delta_{g_t} f + \operatorname{tr}_{g_t} g + \log\frac{\det g_t}{\det h}\right).
\end{aligned}$$

Next, we compute

$$\begin{aligned}
\nabla_i \Delta f =&\ g_\alpha^{\bar{l}k}\nabla_i\nabla_k\nabla_{\bar{l}} f\\
=&\ g_\alpha^{\bar{l}k}\left[\nabla_k\nabla_i\nabla_{\bar{l}} f - g_\alpha^{\bar{q}p} T^\alpha_{ik\bar{q}}\nabla_p\nabla_{\bar{l}} f\right]\\
=&\ g_\alpha^{\bar{l}k}\left[\nabla_k\nabla_i\nabla_{\bar{l}} f - g_\alpha^{\bar{q}p} T^\alpha_{ik\bar{q}}\nabla_p\nabla_{\bar{l}} f\right]\\
=&\ \left[\Delta\partial f - T_\alpha\circ\overline{\partial}\partial f\right]_i.
\end{aligned}$$

Thus plugging this in above and comparing against Lemma 3.6 we conclude that

$$\begin{aligned}
\frac{\partial}{\partial t}\alpha =&\ \Delta_{g_t}\left(\beta - \sqrt{-1}\partial f\right) - T_\alpha\circ\overline{\partial}\left(\beta - \sqrt{-1}\partial f\right) + \sqrt{-1}\operatorname{tr}_{g_t} T + \mu\\
&- \sqrt{-1}\partial\left(\operatorname{tr}_{g_t} g + \log\frac{\det g_t}{\det h}\right).
\end{aligned}$$

It follows that the 1-parameter family of metrics $\omega_\alpha := \omega + \partial\alpha + \overline{\partial\alpha}$ is a solution to pluriclosed flow with the given initial condition. The proposition follows. ☐

4. Torsion potential evolution equations

In this section we derive evolution equations and estimates for the torsion potential along a solution to the 1-form pluriclosed flow. As we will see below, a miraculous cancellation of nonlinear terms occurs which allows for a very clean estimate of the torsion potential. At the end of the section we specialize these estimates to the case when one has a special background metric, which allows for even stronger estimates. These play a crucial role in the proofs of Theorem 1.1 and 1.2.

Definition 4.1. Let (M^{2n}, ω, J) be a complex manifold with pluriclosed metric. Given $\alpha \in \mathcal{H}_\omega$, we say that $\mu \in \Lambda^{2,0}$ is a *torsion potential for ω_α* if

$$T_\alpha = \partial\omega - \overline{\partial}\mu.$$

Observe that $\partial\alpha \in \Lambda^{2,0}$ is a torsion potential for ω_α. We say that $\phi \in \Lambda^{1,0}$ is a *torsion pluripotential for ω_α* if

$$T_\alpha = \partial\omega - \overline{\partial}\partial\phi.$$

Again, one notes that α is a torsion pluripotential for ω_α.

Remark 4.2. The reason for the terminology is that the torsion of the Chern connection associated to a Hermitian metric is $\partial\omega$. In the case of metrics in $\mathcal{H}_\omega$ this reduces to $\partial\omega - \overline{\partial}\partial\overline{\alpha}$. Thus the quantity $\partial\overline{\alpha}$ governs the torsion tensor, up to a background term of $\partial\omega$. From this perspective, we see that the reduction to the 1-form equation has allowed us to get some control over the torsion, a crucial quantity to control in obtaining estimates, from a quantity which has one fewer derivative. As we will see this quantity satisfies a particularly nice evolution equation.

4.1. **Evolution of torsion pluripotentials.** In this subsection we analyze the evolution of the torsion pluripotential along a solution to pluriclosed flow. A crucial input comes from a cancellation in the application of a Bochner formula observed in [35], which we record below.

Lemma 4.3. *([35] Lemma 4.7) Let (M^{2n}, g_t, J) be a solution to pluriclosed flow, and suppose $\beta_t \in \Lambda^{p,0}$ is a one-parameter family satisfying*

$$\frac{\partial}{\partial t}\beta = \Delta_{g_t}\beta + \mu,$$

where $\mu_t \in \Lambda^{p,0}$. Then

$$\frac{\partial}{\partial t}\left|\beta\right|^2 = \Delta\left|\beta\right|^2 - \left|\nabla\beta\right|^2 - \left|\overline{\nabla}\beta\right|^2 - p\left\langle Q, \operatorname{tr}_g\left(\beta\otimes\overline{\beta}\right)\right\rangle + 2\Re\left\langle\beta,\mu\right\rangle. \tag{4.1}$$

Proposition 4.4. *Let (M^{2n}, g_t, J) be a solution to the pluriclosed flow. Given choices $\hat{g}_t, h, \mu$ as above, suppose (β_t, f_t) is a solution to (3.4). Then*

$$\frac{\partial}{\partial t}\left|\beta\right|^2 = \Delta\left|\beta\right|^2 - \left|\nabla\beta\right|^2 - \left|\overline{\nabla}\beta\right|^2 - \left\langle Q, \beta\otimes\overline{\beta}\right\rangle + 2\Re\left\langle\beta, T^{\alpha}\circ\partial\beta + \operatorname{tr}_{g_\alpha} T\right\rangle.$$

Proof. This follows directly from Proposition 3.9 and Lemma 4.3. □

By exploiting some special identities in dimension 4 we can obtain an improved estimate of the evolution of $\left|\beta\right|^2$.

Corollary 4.5. *Let (M^4, g_t, J) be a solution to the pluriclosed flow. Given choices $\hat{g}_t, h, \mu$ as above, suppose (β_t, f_t) is a solution to (3.4). Then*

$$\frac{\partial}{\partial t}\left|\beta\right|^2 \leq \Delta\left|\beta\right|^2 - \left|\overline{\nabla}\beta\right|^2 + 2\Re\left\langle\beta, \operatorname{tr}_{g_\alpha} T\right\rangle.$$

Proof. We adapt the result of Proposition 4.4. Since we are in real dimension 4, ([36] Lemma 4.4) implies that $Q = \frac{1}{2}\left|T\right|^2 g$, thus $-\left\langle Q, \beta\otimes\overline{\beta}\right\rangle = -\frac{1}{2}\left|T\right|^2\left|\beta\right|^2$. Also note that, in a unitary frame for g_α, using the skew-symmetry of T one has

$$\left|T\right|^2 = \sum_{i,j,k=1}^{2} T_{ij\overline{k}}T_{\overline{i}\overline{j}k} = 2(\left|T_{12\overline{1}}\right|^2 + \left|T_{12\overline{2}}\right|^2).$$

Let us now analyze the term $\Re \langle \beta, T^{\alpha} \circ \partial\beta \rangle$. Working again in a unitary frame for g_{α} we see

$$\begin{aligned}
\langle \beta, T^{\alpha} \circ \partial\beta \rangle =& \sum_{i,j,l=1}^{2} T^{\alpha}_{\overline{ij}l} \nabla_i \beta_{\overline{l}} \beta_j \\
=& \beta_1 \sum_{l=1}^{2} T^{\alpha}_{\overline{21}l} \nabla_2 \beta_{\overline{l}} + \beta_2 \sum_{l=1}^{2} T^{\alpha}_{\overline{12}l} \nabla_1 \beta_{\overline{l}} \\
=& \beta_1 T^{\alpha}_{\overline{21}1} \nabla_2 \beta_{\overline{1}} + \beta_1 T^{\alpha}_{\overline{21}2} \nabla_2 \beta_{\overline{2}} + \beta_2 T^{\alpha}_{\overline{12}1} \nabla_1 \beta_{\overline{1}} + \beta_2 T^{\alpha}_{\overline{12}2} \nabla_1 \beta_{\overline{2}} \\
\leq& \frac{1}{2} \Big[\left|\beta_1\right|^2 \left|T^{\alpha}_{\overline{21}1}\right|^2 + \left|\nabla_2 \beta_{\overline{1}}\right|^2 + \left|\beta_1\right|^2 \left|T^{\alpha}_{\overline{21}2}\right|^2 + \left|\nabla_2 \beta_{\overline{2}}\right|^2 \\
&\qquad + \left|\beta_2\right|^2 \left|T^{\alpha}_{\overline{12}1}\right|^2 + \left|\nabla_1 \beta_{\overline{1}}\right|^2 + \left|\beta_2\right|^2 \left|T^{\alpha}_{\overline{12}2}\right|^2 + \left|\nabla_1 \beta_{\overline{2}}\right|^2 \Big] \\
=& \frac{1}{2} \left[\left(\left|\beta_1\right|^2 + \left|\beta_2\right|^2 \right) \left(\left|T^{\alpha}_{\overline{21}1}\right|^2 + \left|T^{\alpha}_{\overline{21}2}\right|^2 \right) + \left|\nabla\beta\right|^2 \right] \\
=& \frac{1}{2} \left|\nabla\beta\right|^2 + \frac{1}{4} \left|T\right|^2 \left|\beta\right|^2 .
\end{aligned}$$

Collecting the above discussion it follows that

$$- \left|\nabla\beta\right| + 2\Re \langle \beta, T^{\alpha} \circ \partial\beta \rangle - \left\langle Q, \beta \otimes \overline{\beta} \right\rangle \leq 0.$$

The result follows. □

4.2. **Evolution of the torsion potential.** In this subsection we derive the evolution of the torsion potential along a solution to the 1-form reduced pluriclosed flow. We begin with a preliminary calculation of $\Delta_{g_{\alpha}} \partial\alpha$.

Lemma 4.6. *In local complex coordinates we have*

$$\begin{aligned}
\left[\Delta_{g_{\alpha}} \partial\alpha\right]_{ij} =& g_{\alpha}^{\overline{q}p} \left[(\partial\alpha)_{ij,\overline{q}p} - g_{\alpha}^{\overline{s}r} \left[g_{i\overline{s},p} (\partial\alpha)_{rj,\overline{q}} + g_{j\overline{s},p} (\partial\alpha)_{ir,\overline{q}} \right] \right] \\
&+ \sqrt{-1} g_{\alpha}^{\overline{q}p} g_{\alpha}^{\overline{s}r} \left[\alpha_{i,\overline{s}p} \alpha_{r,j\overline{q}} + \alpha_{\overline{s},ip} \alpha_{j,r\overline{q}} - \alpha_{\overline{s},ip} \alpha_{r,j\overline{q}} - \alpha_{j,\overline{s}p} \alpha_{r,i\overline{q}} + \alpha_{\overline{s},jp} \alpha_{r,i\overline{q}} - \alpha_{\overline{s},jp} \alpha_{i,r\overline{q}} \right].
\end{aligned}$$

Proof. We directly compute

$$\begin{aligned}
\left[\Delta_{g_{\alpha}} \partial\alpha\right]_{ij} =& g_{\alpha}^{\overline{q}p} \left[\nabla \overline{\nabla} \partial\alpha \right]_{p\overline{q}ij} \\
=& g_{\alpha}^{\overline{q}p} \left[(\partial\alpha)_{ij,\overline{q}p} - \Gamma_{pi}^{r} (\partial\alpha)_{rj,\overline{q}} - \Gamma_{pj}^{r} (\partial\alpha)_{ir,\overline{q}} \right] \\
=& g_{\alpha}^{\overline{q}p} \left[(\partial\alpha)_{ij,\overline{q}p} - g_{\alpha}^{\overline{s}r} (g_{\alpha})_{i\overline{s},p} (\partial\alpha)_{rj,\overline{q}} - g_{\alpha}^{\overline{s}r} (g_{\alpha})_{j\overline{s},p} (\partial\alpha)_{ir,\overline{q}} \right] \\
=& g_{\alpha}^{\overline{q}p} \Big[(\partial\alpha)_{ij,\overline{q}p} - g_{\alpha}^{\overline{s}r} \Big[\left[g_{i\overline{s},p} + \sqrt{-1} \left(\alpha_{i,\overline{s}p} - \alpha_{\overline{s},ip} \right) \right] \left(\alpha_{j,r\overline{q}} - \alpha_{r,j\overline{q}} \right) \\
&\qquad + \left[g_{j\overline{s},p} + \sqrt{-1} \left(\alpha_{j,\overline{s}p} - \alpha_{\overline{s},jp} \right) \right] \left(\alpha_{r,i\overline{q}} - \alpha_{i,r\overline{q}} \right) \Big] \Big] \\
=& g_{\alpha}^{\overline{q}p} \left[(\partial\alpha)_{ij,\overline{q}p} - g_{\alpha}^{\overline{s}r} \left[g_{i\overline{s},p} (\partial\alpha)_{rj,\overline{q}} + g_{j\overline{s},p} (\partial\alpha)_{ir,\overline{q}} \right] \right] \\
&- \sqrt{-1} g_{\alpha}^{\overline{q}p} g_{\alpha}^{\overline{s}r} \big[\alpha_{i,\overline{s}p} \alpha_{j,r\overline{q}} - \alpha_{i,\overline{s}p} \alpha_{r,j\overline{q}} - \alpha_{\overline{s},ip} \alpha_{j,r\overline{q}} + \alpha_{\overline{s},ip} \alpha_{r,j\overline{q}} \\
&\qquad + \alpha_{j,\overline{s}p} \alpha_{r,i\overline{q}} - \alpha_{j,\overline{s}p} \alpha_{i,r\overline{q}} - \alpha_{\overline{s},jp} \alpha_{r,i\overline{q}} + \alpha_{\overline{s},jp} \alpha_{i,r\overline{q}} \big] \\
=& g_{\alpha}^{\overline{q}p} \left[(\partial\alpha)_{ij,\overline{q}p} - g_{\alpha}^{\overline{s}r} \left[g_{i\overline{s},p} (\partial\alpha)_{rj,\overline{q}} + g_{j\overline{s},p} (\partial\alpha)_{ir,\overline{q}} \right] \right] \\
&+ \sqrt{-1} g_{\alpha}^{\overline{q}p} g_{\alpha}^{\overline{s}r} \left[\alpha_{i,\overline{s}p} \alpha_{r,j\overline{q}} + \alpha_{\overline{s},ip} \alpha_{j,r\overline{q}} - \alpha_{\overline{s},ip} \alpha_{r,j\overline{q}} - \alpha_{j,\overline{s}p} \alpha_{r,i\overline{q}} + \alpha_{\overline{s},jp} \alpha_{r,i\overline{q}} - \alpha_{\overline{s},jp} \alpha_{i,r\overline{q}} \right].
\end{aligned}$$

□

Before computing the next evolution equation we record a general coordinate formula.

Lemma 4.7. *Let (M^{2n}, g, J) be a Hermitian manifold. In local complex coordinates we have*

$$(\partial\overline{\partial}_g^*\omega)_{ij} = \sqrt{-1}\left[g^{\overline{q}p}\left(g_{i\overline{q},pj} - g_{j\overline{q},pi}\right) + g^{\overline{q}r}g^{\overline{m}p}\left[g_{r\overline{m},i}g_{j\overline{q},p} - g_{r\overline{m},j}g_{i\overline{q},p}\right]\right].$$

Proof. First we note the basic coordinate calculation

$$\left(\overline{\partial}_g^*\omega\right)_j = \sqrt{-1}g^{\overline{q}p}\left[g_{p\overline{q},j} - g_{j\overline{q},p}\right].$$

Using this we compute

$$\begin{aligned}
\left[\partial\overline{\partial}_g^*\omega\right]_{ij} &= \partial_i\left(\overline{\partial}_g^*\omega\right)_j - \partial_j\left(\overline{\partial}_g^*\omega\right)_i \\
&= \sqrt{-1}\partial_i\left[g^{\overline{q}p}(g_{p\overline{q},j} - g_{j\overline{q},p})\right] - \sqrt{-1}\partial_j\left[g^{\overline{q}p}(g_{p\overline{q},i} - g_{i\overline{q},p})\right] \\
&= \sqrt{-1}\left[g^{\overline{q}p}\left(g_{p\overline{q},ji} - g_{j\overline{q},pi} - g_{p\overline{q},ij} + g_{i\overline{q},pj}\right)\right. \\
&\qquad \left.+g^{\overline{q}r}g^{\overline{m}p}g_{r\overline{m},i}\left(g_{j\overline{q},p} - g_{p\overline{q},j}\right) + g^{\overline{q}r}g^{\overline{m}p}g_{r\overline{m},j}\left(g_{p\overline{q},i} - g_{i\overline{q},p}\right)\right] \\
&= \sqrt{-1}\left[g^{\overline{q}p}\left(g_{i\overline{q},pj} - g_{j\overline{q},pi}\right) + g^{\overline{q}r}g^{\overline{m}p}\left[g_{r\overline{m},i}\left(g_{j\overline{q},p} - g_{p\overline{q},j}\right) + g_{r\overline{m},j}\left(g_{p\overline{q},i} - g_{i\overline{q},p}\right)\right]\right] \\
&= \sqrt{-1}\left[g^{\overline{q}p}\left(g_{i\overline{q},pj} - g_{j\overline{q},pi}\right) + g^{\overline{q}r}g^{\overline{m}p}\left[g_{r\overline{m},i}g_{j\overline{q},p} - g_{r\overline{m},j}g_{i\overline{q},p}\right]\right].
\end{aligned}$$

□

Lemma 4.8. *Given (M^{2n}, g, J), a Hermitian metric h, $\mu \in \Lambda^{1,0}$, and $\alpha \in \mathcal{H}_\omega$, one has*

$$\partial\Psi(g,h,\mu,\alpha) = \Delta_{g_\alpha}\partial\overline{\alpha} - \operatorname{tr}_{g_\alpha}\nabla^\alpha T_g + \partial\mu.$$

Proof. Starting from Lemma 4.7 and plugging in (2.3) yields

$$\begin{aligned}
&(\partial\Psi(g,h,0,\alpha))_{ij} = \left[\partial\left(\overline{\partial}_{g_\alpha}^*\omega_\alpha) - \frac{\sqrt{-1}}{2}\partial\log\frac{\det g_\alpha}{\det h}\right)\right]_{ij} = \left(\partial\overline{\partial}_{g_\alpha}^*\omega_\alpha\right)_{ij} \\
&= \sqrt{-1}\left[g_\alpha^{\overline{q}p}\left(g_{i\overline{q},pj} + \sqrt{-1}(\alpha_{i,\overline{q}pj} - \alpha_{\overline{q},ipj}) - g_{j\overline{q},pi} - \sqrt{-1}(\alpha_{j,\overline{q}pi} - \alpha_{\overline{q},jpi})\right)\right. \\
&\qquad + g_\alpha^{\overline{q}r}g_\alpha^{\overline{m}p}\left[\left(g_{r\overline{m},i} + \sqrt{-1}\left(\alpha_{r,\overline{m}i} - \alpha_{\overline{m},ri}\right)\right)\left(g_{j\overline{q},p} + \sqrt{-1}(\alpha_{j,\overline{q}p} - \alpha_{\overline{q},pj})\right)\right. \\
&\qquad\qquad \left.\left.- \left(g_{r\overline{m},j} + \sqrt{-1}(\alpha_{r,\overline{m}j} - \alpha_{\overline{m},rj})\right)\left(g_{i\overline{q},p} + \sqrt{-1}(\alpha_{i,\overline{q}p} - \alpha_{\overline{q},ip})\right)\right]\right] \\
&= \sqrt{-1}\left[-\sqrt{-1}g_\alpha^{\overline{q}p}\left(\partial\alpha\right)_{ij,\overline{q}p} + g_\alpha^{\overline{q}p}\left(g_{i\overline{q},pj} - g_{j\overline{q},pi}\right)\right. \\
&\qquad + g_\alpha^{\overline{q}r}g_\alpha^{\overline{m}p}\left[g_{r\overline{m},i}g_{j\overline{q},p} + \sqrt{-1}g_{r\overline{m},i}(\alpha_{j,\overline{q}p} - \alpha_{\overline{q},pj}) + \sqrt{-1}g_{j\overline{q},p}(\alpha_{r,\overline{m}i} - \alpha_{\overline{m},ri})\right. \\
&\qquad\qquad - g_{r\overline{m},j}g_{i\overline{q},p} - \sqrt{-1}g_{r\overline{m},j}(\alpha_{i,\overline{q}p} - \alpha_{\overline{q},ip}) - \sqrt{-1}g_{i\overline{q},p}(\alpha_{r,\overline{m}j} - \alpha_{\overline{m},rj}) \\
&\qquad\qquad - \alpha_{r,\overline{m}i}\alpha_{j,\overline{q}p} + \alpha_{r,\overline{m}i}\alpha_{\overline{q},pj} + \alpha_{\overline{m},ri}\alpha_{j,\overline{q}p} - \alpha_{\overline{m},ri}\alpha_{\overline{q},pj} \\
&\qquad\qquad \left.\left.+\alpha_{r,\overline{m}j}\alpha_{i,\overline{q}p} - \alpha_{r,\overline{m}j}\alpha_{\overline{q},ip} - \alpha_{\overline{m},rj}\alpha_{i,\overline{q}p} + \alpha_{\overline{m},rj}\alpha_{\overline{q},ip}\right]\right] \\
&= g_\alpha^{\overline{q}p}\left(\partial\alpha\right)_{ij,\overline{q}p} \\
&\quad + \sqrt{-1}g_\alpha^{\overline{q}r}g_\alpha^{\overline{m}p}\left[\alpha_{r,\overline{m}j}\alpha_{i,\overline{q}p} + \alpha_{\overline{m},ri}\alpha_{j,\overline{q}p} - \alpha_{r,\overline{m}j}\alpha_{\overline{q},ip} - \alpha_{r,\overline{m}i}\alpha_{j,\overline{q}p} + \alpha_{r,\overline{m}i}\alpha_{\overline{q},pj} - \alpha_{\overline{m},rj}\alpha_{i,\overline{q}p}\right] \\
&\quad + \sqrt{-1}\left[g_\alpha^{\overline{q}p}\left(g_{i\overline{q},pj} - g_{j\overline{q},pi}\right)\right] \\
&\quad + \sqrt{-1}g_\alpha^{\overline{q}r}g_\alpha^{\overline{m}p}\left[g_{r\overline{m},i}g_{j\overline{q},p} + \sqrt{-1}g_{r\overline{m},i}(\alpha_{j,\overline{q}p} - \alpha_{\overline{q},pj}) + \sqrt{-1}g_{j\overline{q},p}(\alpha_{r,\overline{m}i} - \alpha_{\overline{m},ri})\right. \\
&\qquad\qquad \left.-g_{r\overline{m},j}g_{i\overline{q},p} - \sqrt{-1}g_{r\overline{m},j}(\alpha_{i,\overline{q}p} - \alpha_{\overline{q},ip}) - \sqrt{-1}g_{i\overline{q},p}(\alpha_{r,\overline{m}j} - \alpha_{\overline{m},rj})\right].
\end{aligned}$$

Now comparing against the result of Lemma 4.6 yields

$$\begin{aligned}
&(\partial\Psi(g,h,0,\alpha))_{ij}\\
=\ & [\Delta_{g_\alpha}\partial\alpha]_{ij} + g_\alpha^{\overline{q}p} g_\alpha^{\overline{s}r}\left[g_{i\overline{s},p}(\partial\alpha)_{rj,\overline{q}} + g_{j\overline{s},p}(\partial\alpha)_{ir,\overline{q}}\right] + \sqrt{-1}\left[g_\alpha^{\overline{q}p}\left(g_{i\overline{q},pj} - g_{j\overline{q},pi}\right)\right]\\
&+ \sqrt{-1} g_\alpha^{\overline{q}r} g_\alpha^{\overline{m}p}\left[g_{r\overline{m},i} g_{j\overline{q},p} + \sqrt{-1} g_{r\overline{m},i}(\alpha_{j,\overline{q}p} - \alpha_{\overline{q},pj}) + \sqrt{-1} g_{j\overline{q},p}(\alpha_{r,\overline{m}i} - \alpha_{\overline{m},ri})\right.\\
&\qquad \left. - g_{r\overline{m},j} g_{i\overline{q},p} - \sqrt{-1} g_{r\overline{m},j}(\alpha_{i,\overline{q}p} - \alpha_{\overline{q},ip}) - \sqrt{-1} g_{i\overline{q},p}(\alpha_{r,\overline{m}j} - \alpha_{\overline{m},rj})\right].
\end{aligned}$$

It remains to identify the lower order terms. First, we relabel indices and combine three of the terms to yield

$$\begin{aligned}
&\sqrt{-1} g_\alpha^{\overline{q}p}\left[g_{i\overline{q},pj} - g_{j\overline{q},pi} + g_\alpha^{\overline{m}l}(g_{p\overline{m},i} g_{j\overline{q},l} - g_{p\overline{m},j} g_{i\overline{q},l})\right]\\
&\quad = \sqrt{-1} g_\alpha^{\overline{q}p}\left[\partial_p(-\sqrt{-1} T_{ji\overline{q}}) + g_\alpha^{\overline{m}l}(g_{p\overline{m},i} g_{j\overline{q},l} - g_{p\overline{m},j} g_{i\overline{q},l})\right]\\
&\quad = \sqrt{-1} g_\alpha^{\overline{q}p}\left[\nabla_p^\alpha(-\sqrt{-1} T_{ji\overline{q}}) + (\Gamma^\alpha)_{pj}^l(-\sqrt{-1} T_{li\overline{q}}) + (\Gamma^\alpha)_{pi}^l(-\sqrt{-1} T_{jl\overline{q}}) + g_\alpha^{\overline{m}l}(g_{p\overline{m},i} g_{j\overline{q},l} - g_{p\overline{m},j} g_{i\overline{q},l})\right]\\
&\quad = \sqrt{-1} g_\alpha^{\overline{q}p}\left[\nabla_p^\alpha(-\sqrt{-1} T_{ji\overline{q}}) + g_\alpha^{\overline{m}l}\left(g_{j\overline{m},p} + \sqrt{-1}\partial_p(\alpha_{j,\overline{m}} - \alpha_{\overline{m},j})\right)(g_{i\overline{q},l} - g_{l\overline{q},i})\right.\\
&\qquad\qquad + g_\alpha^{\overline{m}l}\left(g_{i\overline{m},p} + \sqrt{-1}\partial_p(\alpha_{i,\overline{m}} - \alpha_{\overline{m},i})\right)(g_{l\overline{q},j} - g_{j\overline{q},l})\\
&\qquad\qquad \left. + g_\alpha^{\overline{m}l}(g_{p\overline{m},i} g_{j\overline{q},l} - g_{p\overline{m},j} g_{i\overline{q},l})\right]\\
&\quad = g_\alpha^{\overline{q}p}\left[-\nabla_p^\alpha T_{ij\overline{q}} - g_\alpha^{\overline{m}l}\left((\alpha_{j,\overline{m}p} - \alpha_{\overline{m},jp})(g_{i\overline{q},l} - g_{l\overline{q},i}) + (\alpha_{i,\overline{m}p} - \alpha_{\overline{m},ip})(g_{l\overline{q},j} - g_{j\overline{q},l})\right)\right].
\end{aligned}$$

Inserting this identity into the calculation above, it remains to identify the terms of type $\partial g \star \partial^2\alpha$. These are, after relabeling indices,

$$\begin{aligned}
\partial g \star \partial^2\alpha =\ & g_\alpha^{\overline{q}p} g_\alpha^{\overline{m}l}\left[g_{i\overline{m},p}(\alpha_{j,l\overline{q}} - \alpha_{l,j\overline{q}}) + g_{j\overline{m},p}(\alpha_{l,i\overline{q}} - \alpha_{i,l\overline{q}}) - g_{p\overline{m},i}(\alpha_{j,\overline{q}l} - \alpha_{\overline{q},lj})\right.\\
&\quad - g_{j\overline{q},l}(\alpha_{p,\overline{m}i} - \alpha_{\overline{m},pi}) + g_{p\overline{m},j}(\alpha_{i,\overline{q}l} - \alpha_{\overline{q},il}) + g_{i\overline{q},l}(\alpha_{p,\overline{m}j} - \alpha_{\overline{m},pj})\\
&\quad \left. - (\alpha_{j,\overline{m}p} - \alpha_{\overline{m},jp})(g_{i\overline{q},l} - g_{l\overline{q},i}) - (\alpha_{i,\overline{m}p} - \alpha_{\overline{m},ip})(g_{l\overline{q},j} - g_{j\overline{q},l})\right]\\
=\ & g_\alpha^{\overline{q}p} g_\alpha^{\overline{m}l}\left[g_{i\overline{m},p}\alpha_{j,l\overline{q}} - g_{i\overline{m},p}\alpha_{l,j\overline{q}} + g_{j\overline{m},p}\alpha_{l,i\overline{q}} - g_{j\overline{m},p}\alpha_{i,l\overline{q}} - g_{p\overline{m},i}\alpha_{j,\overline{q}l} + g_{p\overline{m},i}\alpha_{\overline{q},lj}\right.\\
&\quad - g_{j\overline{q},l}\alpha_{p,\overline{m}i} + g_{j\overline{q},l}\alpha_{\overline{m},pi} + g_{p\overline{m},j}\alpha_{i,\overline{q}l} - g_{p\overline{m},j}\alpha_{\overline{q},il} + g_{i\overline{q},l}\alpha_{p,\overline{m}j} - g_{i\overline{q},l}\alpha_{\overline{m},pj}\\
&\quad - g_{i\overline{q},l}\alpha_{j,\overline{m}p} + g_{l\overline{q},i}\alpha_{j,\overline{m}p} + g_{i\overline{q},l}\alpha_{\overline{m},jp} - g_{l\overline{q},i}\alpha_{\overline{m},jp}\\
&\quad \left. - g_{l\overline{q},j}\alpha_{i,\overline{m}p} + g_{j\overline{q},l}\alpha_{i,\overline{m}p} + g_{l\overline{q},j}\alpha_{\overline{m},ip} - g_{j\overline{q},l}\alpha_{\overline{m},ip}\right]\\
=\ & \sum_{i=1}^{20} A_i\\
=\ & 0,
\end{aligned}$$

where the penultimate line defines the terms A_i in the order they appear, and the final line follows from the cancellations $A_1 + A_{13} = A_2 + A_{11} = A_3 + A_7 = A_4 + A_{18} = A_5 + A_{14} = A_6 + A_{16} = A_8 + A_{20} = A_9 + A_{17} = A_{10} + A_{19} = A_{12} + A_{15} = 0$. The lemma follows. ☐

Proposition 4.9. *Let (M^{2n}, g_t, J) be a solution to pluriclosed flow. Fix background data $\hat{g}_t, h, \mu$ and a solution α_t to (3.2). Then*

$$\begin{aligned}
\frac{\partial}{\partial t}\partial\alpha &= \Delta_{g_\alpha}\partial\alpha - \operatorname{tr}_{g_\alpha}\nabla^{g_\alpha} T_{\hat{g}} + \partial\mu,\\
\frac{\partial}{\partial t}\left|\partial\alpha\right|_{g_\alpha}^2 &= \Delta_{g_\alpha}\left|\partial\alpha\right|^2 - \left|\nabla\partial\alpha\right|^2 - \left|\overline{\nabla}\partial\alpha\right|^2 - 2\left\langle Q, \operatorname{tr}\partial\alpha\otimes\overline{\partial\alpha}\right\rangle - 2\Re\left\langle \operatorname{tr}_{g_\alpha}\nabla^{g_\alpha} T_{\hat{g}} + \partial\mu, \overline{\partial\alpha}\right\rangle.
\end{aligned}$$

Proof. The first equation follows directly from Lemma 4.8. The second equation follows from the first and Lemma 4.3. ☐

Proposition 4.10. *Let (M^{2n}, g_t, J) be a solution to pluriclosed flow. Fix background data $\hat{g}_t, h, \mu = 0$ and a solution α_t to (3.2). Suppose furthermore that*

$$\partial \hat{\omega}_t = \partial \hat{\omega}_0 = \overline{\partial} \eta.$$

Let $\phi = \partial \alpha - \eta$. Then

$$\left(\frac{\partial}{\partial t} - \Delta_{g_t} \right) \phi = 0,$$
$$\left(\frac{\partial}{\partial t} - \Delta_{g_t} \right) |\phi|^2 = -|\nabla \phi| - |T_{g_t}|^2 - 2 \left\langle Q, \phi \otimes \overline{\phi} \right\rangle .$$

Proof. We observe that

$$\left(\operatorname{tr}_{g_\alpha} \nabla^{g_\alpha} T_{\hat{g}}\right)_{ij} = g_\alpha^{\overline{q}p} \nabla_p \partial \hat{\omega}_{ij\overline{q}} = g_\alpha^{\overline{q}p} \nabla_p \nabla_{\overline{q}} \eta_{ij} = \Delta_{g_\alpha} \eta_{ij}.$$

Thus using Proposition 4.9 and the assumption that $\mu = 0$ we obtain

$$\frac{\partial}{\partial t} \phi = \frac{\partial}{\partial t} \partial \alpha = \Delta_{g_t} \partial \alpha - \operatorname{tr}_g \nabla^g T_{\hat{g}} = \Delta_{g_t} \left(\partial \alpha - \eta \right) = \Delta_{g_t} \phi.$$

This yields the first claimed equation, and then the second follows from Lemma 4.3 and the fact that

$$\overline{\nabla} \phi = \overline{\partial} \phi = \overline{\partial} \left(\partial \alpha - \eta \right) = -T_{g_t}.$$

□

5. Evans-Krylov Regularity

In this section we prove Theorem 1.7. As explained in the introduction, the idea is to consider a special matrix W defined using a nonlinear combination of first derivatives of α (see Definition 5.2). The matrix W has two crucial properties which are used in Theorem 1.7. First, for any choice of α, W has unit determinant (Lemma 5.4). Second, $W(\alpha_t)$ is a matrix subsolution to a linear uniformly parabolic equation along the pluriclosed flow. This is proved in Proposition 5.11 after a long series of tedious calculations. Using these two crucial properties we establish Theorem 1.7 by adapting the method of Evans-Krylov.

5.1. **Setup.** In analogy with the cone of positivity of Definition 2.2, we define a class of $(1, 0)$-forms which corresponds to uniformly parabolic solutions of pluriclosed flow.

Definition 5.1. Given a domain $U \times [a, b] \subset \mathbb{C}^n \times \mathbb{R}$, let ω denote the standard flat Kähler form on $\mathbb{C}^n$ and let

$$\mathcal{E}_U^{\lambda,\Lambda} = \{\alpha : [a, b] \to \Gamma(\Lambda^{1,0}(U)) \mid \forall t \in [a, b],\ \lambda\omega \leq \sqrt{-1} \left(\overline{\partial} \alpha_t + \partial \overline{\alpha_t} \right) \leq \Lambda \omega,\ |\partial \alpha|^2 \leq \Lambda\}.$$

Moreover, given $\alpha \in \mathcal{E}_U^{\lambda,\Lambda}$, let $\omega_\alpha = \sqrt{-1} \left(\overline{\partial} \alpha + \partial \overline{\alpha} \right)$, with corresponding metric coefficients

$$g^\alpha_{i\overline{j}} = \sqrt{-1} \left(\alpha_{i,\overline{j}} - \alpha_{\overline{j},i} \right). \tag{5.1}$$

When some α is given we will frequently drop the dependence of g on α in the notation. At various points for notational simplicity we set $\beta_{ij} = \sqrt{-1} \partial \alpha_{ij}, \beta_i^{\overline{j}} = g^{\overline{j}k} \beta_{ik}$.

Definition 5.2. Given $\alpha \in \mathcal{E}_U^{\lambda\Lambda}$, let

$$W = \begin{pmatrix} g_{i\overline{j}} + \partial \alpha_{ip} \overline{\partial \alpha}_{\overline{j}\overline{q}} g^{\overline{q}p} & \sqrt{-1} \partial \alpha_{ip} g^{\overline{l}p} \\ -\sqrt{-1} \overline{\partial \alpha}_{\overline{j}\overline{q}} g^{\overline{q}k} & g^{\overline{l}k} \end{pmatrix} = \begin{pmatrix} g_{i\overline{j}} + \beta_{ik} \overline{\beta}_{\overline{j}\overline{l}} g^{\overline{l}k} & \beta_i^{\overline{l}} \\ \beta_{\overline{j}}^k & g^{\overline{l}k} \end{pmatrix}.$$

Remark 5.3. This matrix W can be interpreted as the natural "Born-Infeld" metric on the split tangent bundle $T \oplus T^*$, where $\partial\alpha$ is playing the role of the skew-symmetric "b-field." This metric first arose through investigations into mathematical physics [33, 45]. Later, investigations into generalized Kähler geometry [21, 18] identified natural geometric interpretations of this object, called a generalized metric on $T \oplus T^*$. Previously in joint work with Tian [38] we had shown that the pluriclosed flow is diffeomorphism-equivalent to the renormalization group flow arising from a nonlinear sigma model coupled to a b-field. Moreover, in [39] the author and Tian exhibited that the pluriclosed flow preserves generalized Kähler geometry in the appropriate sense. These connections between the pluriclosed flow and supersymmetry/generalized Kähler geometry inspired the choice of matrix W, which can be shown to obey a remarkable differential inequality which lies at the heart of Theorem 1.7.

Lemma 5.4. *Given $\alpha \in \Lambda^{1,0}$ such that $\sqrt{-1}\left(\overline{\partial}\alpha + \partial\overline{\alpha}\right) > 0$, one has $\det W(\alpha) = 1$.*

Proof. First recall the block determinant formula

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det D \det(A - BD^{-1}C).$$

Using this we compute

$$\begin{aligned} \det W = & \det g^{-1} \det \left[g_{i\overline{j}} + \partial\alpha_{ip}\overline{\partial\alpha}_{\overline{j}\overline{q}} g^{\overline{q}p} - \left(\sqrt{-1}\partial\alpha_{ip} g^{\overline{l}p} g_{k\overline{l}} (-\sqrt{-1}\overline{\partial\alpha})_{\overline{j}\overline{q}} g^{\overline{q}k}\right)\right] \\ = & \det g^{-1} \det g \\ = & \ 1. \end{aligned}$$

□

Next we derive the evolution equation for α along a solution to the pluriclosed flow in this setting. Using Lemma 3.5 we observe that

$$\Phi(\alpha)_i = g^{\overline{j}k}\left[\alpha_{i,\overline{j}k} - \frac{1}{2}\left(\alpha_{\overline{j},ik} + \alpha_{k,\overline{j}i}\right)\right].$$

Thus in this setting the pluriclosed flow reduces to

$$\frac{\partial}{\partial t}\alpha_i = g^{\overline{j}k}\left[\alpha_{i,\overline{j}k} - \frac{1}{2}\left(\alpha_{\overline{j},ik} + \alpha_{k,\overline{j}i}\right)\right]. \tag{5.2}$$

5.2. **Differential Inequalities.** In this subsection we establish in Proposition 5.11 that along a solution to (5.2), $W(\alpha_t)$ is a matrix subsolution to a uniformly parabolic equation. This fact is central to the proof of Theorem 1.7, and follows from a lengthy calculation broken up into a series of lemmas below.

Lemma 5.5. *Given the setup above,*

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) g_{i\overline{j}} = g^{\overline{l}p} g^{\overline{q}k}\left[\alpha_{p,\overline{q}\overline{j}}\alpha_{i,\overline{l}k} - \alpha_{p,\overline{q}\overline{j}}\alpha_{\overline{l},ik} - \alpha_{\overline{q},p\overline{j}}\alpha_{i,\overline{l}k} + \alpha_{\overline{q},p\overline{j}}\alpha_{k,\overline{l}i} - \alpha_{p,\overline{q}i}\alpha_{\overline{j},k\overline{l}} + \alpha_{\overline{q},pi}\alpha_{\overline{j},k\overline{l}}\right].$$

Proof. We compute

$$\begin{aligned}
\frac{\partial}{\partial t} g_{i\bar{j}} &= \sqrt{-1}\left(\dot{\alpha}_{i,\bar{j}} - \dot{\alpha}_{\bar{j},i}\right) \\
&= \sqrt{-1}\left[g^{\bar{l}k}\left(\alpha_{i,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},ik} + \alpha_{k,\bar{l}i}\right)\right)\right]_{,\bar{j}} - \sqrt{-1}\left[g^{\bar{k}l}\left(\alpha_{\bar{j},l\bar{k}} - \frac{1}{2}\left(\alpha_{l,\bar{j}\bar{k}} + \alpha_{\bar{k},l\bar{j}}\right)\right)\right]_{,i} \\
&= \sqrt{-1} g^{\bar{l}k}\left[\alpha_{i,\bar{l}k\bar{j}} - \alpha_{\bar{j},ik\bar{l}}\right] \\
&\quad - \sqrt{-1} g^{\bar{l}p} g_{p\bar{q},\bar{j}} g^{\bar{q}k}\left[\alpha_{i,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},ik} + \alpha_{k,\bar{l}i}\right)\right] + \sqrt{-1} g^{\bar{k}p} g_{p\bar{q},i} g^{\bar{q}l}\left[\alpha_{\bar{j},l\bar{k}} - \frac{1}{2}\left(\alpha_{l,\bar{j}\bar{k}} + \alpha_{\bar{k},l\bar{j}}\right)\right] \\
&= \sqrt{-1} g^{\bar{l}k}\left[\alpha_{i,\bar{j}k\bar{l}} - \alpha_{\bar{j},ik\bar{l}}\right] + g^{\bar{l}p} g^{\bar{q}k}\left(\alpha_{p,\bar{q}\bar{j}} - \alpha_{\bar{q},p\bar{j}}\right)\left[\alpha_{i,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},ik} + \alpha_{k,\bar{l}i}\right)\right] \\
&\quad - g^{\bar{k}p} g^{\bar{q}l}\left(\alpha_{p,\bar{q}i} - \alpha_{\bar{q},pi}\right)\left[\alpha_{\bar{j},l\bar{k}} - \frac{1}{2}\left(\alpha_{l,\bar{j}\bar{k}} + \alpha_{\bar{k},l\bar{j}}\right)\right] \\
&= \sqrt{-1} g^{\bar{l}k}\left[\alpha_{i,\bar{j}k\bar{l}} - \alpha_{\bar{j},ik\bar{l}}\right] \\
&\quad + g^{\bar{l}p} g^{\bar{q}k}\Big[\alpha_{p,\bar{q}\bar{j}}\alpha_{i,\bar{l}k} - \frac{1}{2}\alpha_{p,\bar{q}\bar{j}}\alpha_{\bar{l},ik} - \frac{1}{2}\alpha_{p,\bar{q}\bar{j}}\alpha_{k,\bar{l}i} - \alpha_{\bar{q},p\bar{j}}\alpha_{i,\bar{l}k} + \frac{1}{2}\alpha_{\bar{q},p\bar{j}}\alpha_{\bar{l},ik} + \frac{1}{2}\alpha_{\bar{q},p\bar{j}}\alpha_{k,\bar{l}i} \\
&\qquad - \alpha_{p,\bar{q}i}\alpha_{\bar{j},k\bar{l}} + \frac{1}{2}\alpha_{p,\bar{q}i}\alpha_{k,\bar{j}\bar{l}} + \frac{1}{2}\alpha_{p,\bar{q}i}\alpha_{\bar{l},k\bar{j}} + \alpha_{\bar{q},pi}\alpha_{\bar{j},k\bar{l}} - \frac{1}{2}\alpha_{\bar{q},pi}\alpha_{k,\bar{j}\bar{l}} - \frac{1}{2}\alpha_{\bar{q},pi}\alpha_{\bar{l},k\bar{j}}\Big] \\
&= \sqrt{-1} g^{\bar{l}k}\left[\alpha_{i,\bar{j}k\bar{l}} - \alpha_{\bar{j},ik\bar{l}}\right] + \sum_{i=1}^{12} A_i.
\end{aligned}$$

We observe that $A_2 = A_{11}, A_3 + A_8 = 0, A_5 + A_{12} = 0, A_6 = A_9$, finishing the result. □

Lemma 5.6. *Given the setup above,*

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) g^{\bar{s}r} = -g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}p} g^{\bar{q}k}\left[\left(\alpha_{\bar{q},p\bar{j}} - \alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i} - \alpha_{i,\bar{l}k}\right) + \left(\alpha_{i,\bar{q}\bar{l}} - \alpha_{\bar{q},i\bar{l}}\right)\left(\alpha_{\bar{j},kp} - \alpha_{k,\bar{j}p}\right)\right].$$

Proof. First we compute using Lemma 5.5,

$$\begin{aligned}
\frac{\partial}{\partial t} g^{\bar{s}r} &= -g^{\bar{s}i}\dot{g}_{i\bar{j}} g^{\bar{j}r} \\
&= -\sqrt{-1} g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}k}\left[\alpha_{i,\bar{j}k\bar{l}} - \alpha_{\bar{j},ik\bar{l}}\right] \\
&\quad - g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}p} g^{\bar{q}k}\left[\alpha_{p,\bar{q}\bar{j}}\alpha_{i,\bar{l}k} - \alpha_{p,\bar{q}\bar{j}}\alpha_{\bar{l},ik} - \alpha_{\bar{q},p\bar{j}}\alpha_{i,\bar{l}k} + \alpha_{\bar{q},p\bar{j}}\alpha_{k,\bar{l}i} - \alpha_{p,\bar{q}i}\alpha_{\bar{j},k\bar{l}} + \alpha_{\bar{q},pi}\alpha_{\bar{j},k\bar{l}}\right].
\end{aligned}$$

Next we observe

$$\begin{aligned}
\Delta_{g_\alpha} g^{\bar{s}r} &= g^{\bar{l}k}\left[g^{\bar{s}r}\right]_{,k\bar{l}}\\
&= g^{\bar{l}k}\left[-g^{\bar{s}i}g_{i\bar{j},k}g^{\bar{j}r}\right]_{,\bar{l}}\\
&= g^{\bar{l}k}\left[g^{\bar{s}a}g_{a\bar{b},\bar{l}}g^{\bar{b}i}g_{i\bar{j},k}g^{\bar{j}r} - g^{\bar{s}i}g_{i\bar{j},k\bar{l}}g^{\bar{j}r} + g^{\bar{s}i}g_{i\bar{j},k}g^{\bar{j}a}g_{a\bar{b},\bar{l}}g^{\bar{b}r}\right]\\
&= -\sqrt{-1}g^{\bar{l}k}g^{\bar{s}i}g^{\bar{r}j}\left[\alpha_{i,\bar{j}k\bar{l}} - \alpha_{\bar{j},ik\bar{l}}\right] - g^{\bar{l}k}g^{\bar{s}a}g^{\bar{b}i}g^{\bar{j}r}\left(\alpha_{a,\bar{b}\bar{l}} - \alpha_{\bar{b},a\bar{l}}\right)\left(\alpha_{i,\bar{j}k} - \alpha_{\bar{j},ik}\right)\\
&\quad - g^{\bar{l}k}g^{\bar{s}i}g^{\bar{j}a}g^{\bar{b}r}\left(\alpha_{i,\bar{j}k} - \alpha_{\bar{j},ik}\right)\left(\alpha_{a,\bar{b}\bar{l}} - \alpha_{\bar{b},a\bar{l}}\right)\\
&= -\sqrt{-1}g^{\bar{l}k}g^{\bar{s}i}g^{\bar{r}j}\left[\alpha_{i,\bar{j}k\bar{l}} - \alpha_{\bar{j},ik\bar{l}}\right]\\
&\quad - g^{\bar{l}k}g^{\bar{s}a}g^{\bar{b}i}g^{\bar{j}r}\left[\alpha_{a,\bar{b}\bar{l}}\alpha_{i,\bar{j}k} - \alpha_{a,\bar{b}\bar{l}}\alpha_{\bar{j},ik} - \alpha_{\bar{b},a\bar{l}}\alpha_{i,\bar{j}k} + \alpha_{\bar{b},a\bar{l}}\alpha_{\bar{j},ik}\right]\\
&\quad - g^{\bar{l}k}g^{\bar{s}i}g^{\bar{j}a}g^{\bar{b}r}\left[\alpha_{i,\bar{j}k}\alpha_{a,\bar{b}\bar{l}} - \alpha_{i,\bar{j}k}\alpha_{\bar{b},a\bar{l}} - \alpha_{\bar{j},ik}\alpha_{a,\bar{b}\bar{l}} + \alpha_{\bar{j},ik}\alpha_{\bar{b},a\bar{l}}\right].
\end{aligned}$$

Combining the two calculations above yields

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) g^{\bar{s}r} =&\ -g^{\bar{s}i}g^{\bar{j}r}g^{\bar{l}p}g^{\bar{q}k}\left[\alpha_{p,\bar{q}\bar{j}}\alpha_{i,\bar{l}k} - \alpha_{p,\bar{q}\bar{j}}\alpha_{\bar{l},ik} - \alpha_{\bar{q},p\bar{j}}\alpha_{i,\bar{l}k} + \alpha_{\bar{q},p\bar{j}}\alpha_{k,\bar{l}i} - \alpha_{p,\bar{q}i}\alpha_{\bar{j},k\bar{l}} + \alpha_{\bar{q},pi}\alpha_{\bar{j},k\bar{l}}\right]\\
&+ g^{\bar{l}k}g^{\bar{s}a}g^{\bar{b}i}g^{\bar{j}r}\left[\alpha_{a,\bar{b}\bar{l}}\alpha_{i,\bar{j}k} - \alpha_{a,\bar{b}\bar{l}}\alpha_{\bar{j},ik} - \alpha_{\bar{b},a\bar{l}}\alpha_{i,\bar{j}k} + \alpha_{\bar{b},a\bar{l}}\alpha_{\bar{j},ik}\right]\\
&+ g^{\bar{l}k}g^{\bar{s}i}g^{\bar{j}a}g^{\bar{b}r}\left[\alpha_{i,\bar{j}k}\alpha_{a,\bar{b}\bar{l}} - \alpha_{i,\bar{j}k}\alpha_{\bar{b},a\bar{l}} - \alpha_{\bar{j},ik}\alpha_{a,\bar{b}\bar{l}} + \alpha_{\bar{j},ik}\alpha_{\bar{b},a\bar{l}}\right]\\
=&\ \sum_{i=1}^{14} A_i.
\end{aligned}$$

We observe the cancellations $A_1 + A_{11} = A_2 + A_{13} = A_6 + A_{14} = 0$, leaving

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) g^{\bar{s}r}\\
&= -g^{\bar{s}i}g^{\bar{j}r}g^{\bar{l}p}g^{\bar{q}k}\left[-\alpha_{\bar{q},p\bar{j}}\alpha_{i,\bar{l}k} + \alpha_{\bar{q},p\bar{j}}\alpha_{k,\bar{l}i} - \alpha_{p,\bar{q}i}\alpha_{\bar{j},k\bar{l}}\right]\\
&\quad + g^{\bar{l}k}g^{\bar{s}a}g^{\bar{b}i}g^{\bar{j}r}\left[\alpha_{a,\bar{b}\bar{l}}\alpha_{i,\bar{j}k} - \alpha_{a,\bar{b}\bar{l}}\alpha_{\bar{j},ik} - \alpha_{\bar{b},a\bar{l}}\alpha_{i,\bar{j}k} + \alpha_{\bar{b},a\bar{l}}\alpha_{\bar{j},ik}\right]\\
&\quad + g^{\bar{l}k}g^{\bar{s}i}g^{\bar{j}a}g^{\bar{b}r}\left[-\alpha_{i,\bar{j}k}\alpha_{\bar{b},a\bar{l}}\right]\\
&= g^{\bar{s}i}g^{\bar{j}r}g^{\bar{l}p}g^{\bar{q}k}\Big[\alpha_{\bar{q},p\bar{j}}\alpha_{i,\bar{l}k} - \alpha_{\bar{q},p\bar{j}}\alpha_{k,\bar{l}i} + \alpha_{p,\bar{q}i}\alpha_{\bar{j},k\bar{l}} + \alpha_{i,\bar{q}\bar{l}}\alpha_{k,\bar{j}p}\\
&\qquad\qquad - \alpha_{i,\bar{q}\bar{l}}\alpha_{\bar{j},kp} - \alpha_{\bar{q},i\bar{l}}\alpha_{k,\bar{j}p} + \alpha_{\bar{q},i\bar{l}}\alpha_{\bar{j},kp} - \alpha_{i,\bar{q}p}\alpha_{\bar{j},k\bar{l}}\Big]\\
&= -g^{\bar{s}i}g^{\bar{j}r}g^{\bar{l}p}g^{\bar{q}k}\left[\left(\alpha_{\bar{q},p\bar{j}} - \alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i} - \alpha_{i,\bar{l}k}\right) + \left(\alpha_{i,\bar{q}\bar{l}} - \alpha_{\bar{q},i\bar{l}}\right)\left(\alpha_{\bar{j},kp} - \alpha_{k,\bar{j}p}\right)\right],
\end{aligned}$$

as claimed. □

Lemma 5.7. *Given the setup above,*

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right)\sqrt{-1}\partial\alpha_{uv} = g^{\bar{l}p}g^{\bar{q}k}\left[\alpha_{p,\bar{q}v}\alpha_{u,\bar{l}k} - \alpha_{p,\bar{q}v}\alpha_{\bar{l},uk} - \alpha_{\bar{q},pv}\alpha_{u,\bar{l}k} + \alpha_{\bar{q},pv}\alpha_{k,\bar{l}u} - \alpha_{p,\bar{q}u}\alpha_{v,\bar{l}k} + \alpha_{\bar{q},pu}\alpha_{v,\bar{l}k}\right],$$

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right)(-\sqrt{-1}\bar{\partial}\overline{\alpha}_{\overline{uv}}) = g^{\bar{l}p}g^{\bar{q}k}\left[\alpha_{p,\bar{q}\bar{u}}\alpha_{\bar{v},\bar{l}k} - \alpha_{p,\bar{q}\bar{u}}\alpha_{\bar{l},k\bar{v}} - \alpha_{\bar{q},p\bar{u}}\alpha_{\bar{v},\bar{l}k} + \alpha_{\bar{q},p\bar{u}}\alpha_{k,\bar{v}\bar{l}} - \alpha_{p,\bar{q}\bar{v}}\alpha_{\bar{u},\bar{l}k} + \alpha_{\bar{q},p\bar{v}}\alpha_{\bar{u},\bar{l}k}\right].$$

Proof. We establish the first equation, with the second following by conjugation. We directly compute

$$
\begin{aligned}
\frac{\partial}{\partial t}\sqrt{-1}\partial\alpha_{uv} &= \sqrt{-1}\dot{\alpha}_{u,v} - \sqrt{-1}\dot{\alpha}_{v,u} \\
&= \sqrt{-1}\left[g^{\bar{l}k}\left(\alpha_{u,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},uk} + \alpha_{k,\bar{l}u}\right)\right)\right]_{,v} - \sqrt{-1}\left[g^{\bar{l}k}\left(\alpha_{v,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},vk} + \alpha_{k,\bar{l}v}\right)\right)\right]_{,u} \\
&= \Delta_{g_\alpha}\sqrt{-1}\partial\alpha_{uv} - \sqrt{-1}g^{\bar{l}p}g_{p\bar{q},v}g^{\bar{q}k}\left(\alpha_{u,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},uk} + \alpha_{k,\bar{l}u}\right)\right) \\
&\quad + \sqrt{-1}g^{\bar{l}p}g_{p\bar{q},u}g^{\bar{q}k}\left(\alpha_{v,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},vk} + \alpha_{k,\bar{l}v}\right)\right) \\
&= \Delta_{g_\alpha}\sqrt{-1}\partial\alpha_{uv} + g^{\bar{l}p}g^{\bar{q}k}\left(\alpha_{p,\bar{q}v} - \alpha_{\bar{q},pv}\right)\left(\alpha_{u,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},uk} + \alpha_{k,\bar{l}u}\right)\right) \\
&\quad - g^{\bar{l}p}g^{\bar{q}k}\left(\alpha_{p,\bar{q}u} - \alpha_{\bar{q},pu}\right)\left(\alpha_{v,\bar{l}k} - \frac{1}{2}\left(\alpha_{\bar{l},vk} + \alpha_{k,\bar{l}v}\right)\right) \\
&= \Delta_{g_\alpha}\sqrt{-1}\partial\alpha_{uv} \\
&\quad + g^{\bar{l}p}g^{\bar{q}k}\Big[\alpha_{p,\bar{q}v}\alpha_{u,\bar{l}k} - \frac{1}{2}\alpha_{p,\bar{q}v}\alpha_{\bar{l},uk} - \frac{1}{2}\alpha_{p,\bar{q}v}\alpha_{k,\bar{l}u} - \alpha_{\bar{q},pv}\alpha_{u,\bar{l}k} + \frac{1}{2}\alpha_{\bar{q},pv}\alpha_{\bar{l},uk} + \frac{1}{2}\alpha_{\bar{q},pv}\alpha_{k,\bar{l}u} \\
&\qquad - \alpha_{p,\bar{q}u}\alpha_{v,\bar{l}k} + \frac{1}{2}\alpha_{p,\bar{q}u}\alpha_{\bar{l},vk} + \frac{1}{2}\alpha_{p,\bar{q}u}\alpha_{k,\bar{l}v} + \alpha_{\bar{q},pu}\alpha_{v,\bar{l}k} - \frac{1}{2}\alpha_{\bar{q},pu}\alpha_{\bar{l},vk} - \frac{1}{2}\alpha_{\bar{q},pu}\alpha_{k,\bar{l}v}\Big] \\
&= \Delta_{g_\alpha}\sqrt{-1}\partial\alpha_{uv} + \sum_{i=1}^{12} A_i.
\end{aligned}
$$

We observe that $A_2 = A_{12}, A_3 + A_9 = 0, A_5 + A_{11} = 0, A_6 = A_8$. The result follows. $\square$

Lemma 5.8. *Given the setup above,*

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right)\sqrt{-1}\partial\alpha_{ur}g^{\bar{s}r} \\
&\qquad = -g^{\bar{l}k}g^{\bar{s}p}g^{\bar{q}r}\left[\left(\alpha_{u,rk} - \alpha_{r,uk}\right) + \beta_u^{\bar{v}}\left(\alpha_{\bar{v},rk} - \alpha_{r,\bar{v}k}\right)\right]\left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right) \\
&\qquad\quad + g^{\bar{l}k}g^{\bar{s}p}g^{\bar{q}r}\left[\left(\alpha_{u,\bar{l}r} - \alpha_{\bar{l},ur}\right) + \beta_u^{\bar{v}}\left(\alpha_{\bar{v},r\bar{l}} - \alpha_{\bar{l},r\bar{v}}\right)\right]\left(\alpha_{k,\bar{q}p} - \alpha_{p,\bar{q}k}\right).
\end{aligned}
$$

Proof. First we compute

$$
\begin{aligned}
\Delta_{g_\alpha}\sqrt{-1}\partial\alpha_{ur}g^{\bar{s}r} &= g^{\bar{l}k}\left(\sqrt{-1}\partial\alpha_{ur}g^{\bar{s}r}\right)_{,k\bar{l}} \\
&= \sqrt{-1}g^{\bar{l}k}\left[(\partial\alpha)_{ur,k\bar{l}}g^{\bar{s}r} + (\partial\alpha)_{ur,k}(g^{\bar{s}r})_{,\bar{l}} + (\partial\alpha)_{ur,\bar{l}}(g^{\bar{s}r})_{,k} + \partial\alpha_{ur}(g^{\bar{s}r})_{,k\bar{l}}\right] \\
&= \Delta_{g_\alpha}\left[\sqrt{-1}\partial\alpha_{ur}\right]g^{\bar{s}r} + \sqrt{-1}\partial\alpha_{ur}\Delta_{g_\alpha}g^{\bar{s}r} \\
&\quad + \sqrt{-1}g^{\bar{l}k}\left[-\left(\alpha_{u,rk} - \alpha_{r,uk}\right)g^{\bar{s}p}g_{p\bar{q},\bar{l}}g^{\bar{q}r} - \left(\alpha_{u,r\bar{l}} - \alpha_{r,u\bar{l}}\right)g^{\bar{s}p}g_{p\bar{q},k}g^{\bar{q}r}\right] \\
&= \Delta_{g_\alpha}\left[\sqrt{-1}\partial\alpha_{ur}\right]g^{\bar{s}r} + \sqrt{-1}\partial\alpha_{ur}\Delta_{g_\alpha}g^{\bar{s}r} \\
&\quad + g^{\bar{l}k}g^{\bar{s}p}g^{\bar{q}r}\left[\left(\alpha_{u,rk} - \alpha_{r,uk}\right)\left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right) + \left(\alpha_{u,r\bar{l}} - \alpha_{r,u\bar{l}}\right)\left(\alpha_{p,\bar{q}k} - \alpha_{\bar{q},pk}\right)\right].
\end{aligned}
$$

Using this in conjunction with Lemmas 5.6, 5.7 we yield

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) \sqrt{-1}\partial \alpha_{ur} g^{\bar{s}r} \\
=&\ \left[\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) \sqrt{-1}\partial \alpha_{ur}\right] g^{\bar{s}r} + \left[\sqrt{-1}\partial \alpha_{ur}\right] \left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) g^{\bar{s}r} \\
&- g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[(\alpha_{u,rk} - \alpha_{r,uk}) \left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right) + \left(\alpha_{u,r\bar{l}} - \alpha_{r,u\bar{l}}\right)(\alpha_{p,\bar{q}k} - \alpha_{\bar{q},pk}) \right] \\
=&\ g^{\bar{l}p} g^{\bar{q}k} g^{\bar{s}r} \left[\alpha_{p,\bar{q}r} \alpha_{u,\bar{l}k} - \alpha_{p,\bar{q}r}\alpha_{\bar{l},uk} - \alpha_{\bar{q},pr}\alpha_{u,\bar{l}k} + \alpha_{\bar{q},pr}\alpha_{k,\bar{l}u} - \alpha_{p,\bar{q}u}\alpha_{r,\bar{l}k} + \alpha_{\bar{q},pu}\alpha_{r,\bar{l}k} \right] \\
&- \left[\sqrt{-1}\partial\alpha\right]_{ur} g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}p} g^{\bar{q}k} \left[\left(\alpha_{\bar{q},p\bar{j}} - \alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i} - \alpha_{i,\bar{l}k}\right) + \left(\alpha_{i,\bar{q}\bar{l}} - \alpha_{\bar{q},i\bar{l}}\right)\left(\alpha_{\bar{j},kp} - \alpha_{k,\bar{j}p}\right) \right] \\
&- g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[(\alpha_{u,rk} - \alpha_{r,uk}) \left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right) + \left(\alpha_{u,r\bar{l}} - \alpha_{r,u\bar{l}}\right)(\alpha_{p,\bar{q}k} - \alpha_{\bar{q},pk}) \right] \\
=&\ - \left[\sqrt{-1}\partial\alpha\right]_{ur} g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}p} g^{\bar{q}k} \left[\left(\alpha_{\bar{q},p\bar{j}} - \alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i} - \alpha_{i,\bar{l}k}\right) + \left(\alpha_{i,\bar{q}\bar{l}} - \alpha_{\bar{q},i\bar{l}}\right)\left(\alpha_{\bar{j},kp} - \alpha_{k,\bar{j}p}\right) \right] \\
&+ g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \Big[\alpha_{k,\bar{q}p}\alpha_{u,\bar{l}r} - \alpha_{k,\bar{q}p}\alpha_{\bar{l},ur} - \alpha_{\bar{q},kp}\alpha_{u,\bar{l}r} + \alpha_{\bar{q},kp}\alpha_{r,\bar{l}u} - \alpha_{k,\bar{q}u}\alpha_{p,\bar{l}r} + \alpha_{\bar{q},ku}\alpha_{p,\bar{l}r} \\
&\qquad - \alpha_{u,rk}\alpha_{p,\bar{q}\bar{l}} + \alpha_{u,rk}\alpha_{\bar{q},p\bar{l}} + \alpha_{r,uk}\alpha_{p,\bar{q}\bar{l}} - \alpha_{r,uk}\alpha_{\bar{q},p\bar{l}} \\
&\qquad - \alpha_{u,r\bar{l}}\alpha_{p,\bar{q}k} + \alpha_{u,r\bar{l}}\alpha_{\bar{q},pk} + \alpha_{r,u\bar{l}}\alpha_{p,\bar{q}k} - \alpha_{r,u\bar{l}}\alpha_{\bar{q},pk} \Big] \\
=&\ - \left[\sqrt{-1}\partial\alpha\right]_{ur} g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}p} g^{\bar{q}k} \left[\left(\alpha_{\bar{q},p\bar{j}} - \alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i} - \alpha_{i,\bar{l}k}\right) + \left(\alpha_{i,\bar{q}\bar{l}} - \alpha_{\bar{q},i\bar{l}}\right)\left(\alpha_{\bar{j},kp} - \alpha_{k,\bar{j}p}\right) \right] \\
&+ \sum_{i=1}^{14} A_i.
\end{aligned}
$$

We observe the cancellations $A_3 + A_{12} = A_4 + A_{14} = A_5 + A_{13} = 0$, and apply further simplifications to yield

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) \sqrt{-1}\partial \alpha_{ur} g^{\bar{s}r} \\
=&\ - \left[\sqrt{-1}\partial\alpha\right]_{ur} g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}p} g^{\bar{q}k} \left[\left(\alpha_{\bar{q},p\bar{j}} - \alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i} - \alpha_{i,\bar{l}k}\right) + \left(\alpha_{i,\bar{q}\bar{l}} - \alpha_{\bar{q},i\bar{l}}\right)\left(\alpha_{\bar{j},kp} - \alpha_{k,\bar{j}p}\right) \right] \\
&+ g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \Big[\alpha_{k,\bar{q}p}\alpha_{u,\bar{l}r} - \alpha_{k,\bar{q}p}\alpha_{\bar{l},ur} + \alpha_{\bar{q},ku}\alpha_{p,\bar{l}r} - \alpha_{u,rk}\alpha_{p,\bar{q}\bar{l}} \\
&\qquad + \alpha_{u,rk}\alpha_{\bar{q},p\bar{l}} + \alpha_{r,uk}\alpha_{p,\bar{q}\bar{l}} - \alpha_{r,uk}\alpha_{\bar{q},p\bar{l}} - \alpha_{u,r\bar{l}}\alpha_{p,\bar{q}k} \Big] \\
=&\ - \beta_u^{\bar{j}} g^{\bar{s}i} g^{\bar{l}p} g^{\bar{q}k} \left[\left(\alpha_{\bar{q},p\bar{j}} - \alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i} - \alpha_{i,\bar{l}k}\right) + \left(\alpha_{i,\bar{q}\bar{l}} - \alpha_{\bar{q},i\bar{l}}\right)\left(\alpha_{\bar{j},kp} - \alpha_{k,\bar{j}p}\right) \right] \\
&+ g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[(\alpha_{r,uk} - \alpha_{u,rk})\left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right) + \left(\alpha_{u,\bar{l}r} - \alpha_{\bar{l},ur}\right)(\alpha_{k,\bar{q}p} - \alpha_{p,\bar{q}k}) \right] \\
=&\ - \beta_u^{\bar{v}} g^{\bar{s}p} g^{\bar{l}k} g^{\bar{q}r} \left[(\alpha_{\bar{q},k\bar{v}} - \alpha_{\bar{v},k\bar{q}})\left(\alpha_{r,\bar{l}p} - \alpha_{p,\bar{l}r}\right) + \left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right)(\alpha_{\bar{v},rk} - \alpha_{r,\bar{v}k}) \right] \\
&+ g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[(\alpha_{r,uk} - \alpha_{u,rk})\left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right) + \left(\alpha_{u,\bar{l}r} - \alpha_{\bar{l},ur}\right)(\alpha_{k,\bar{q}p} - \alpha_{p,\bar{q}k}) \right] \\
=&\ - g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[(\alpha_{u,rk} - \alpha_{r,uk}) + \beta_u^{\bar{v}}(\alpha_{\bar{v},rk} - \alpha_{r,\bar{v}k}) \right] \left(\alpha_{p,\bar{q}\bar{l}} - \alpha_{\bar{q},p\bar{l}}\right) \\
&+ g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[\left(\alpha_{u,\bar{l}r} - \alpha_{\bar{l},ur}\right) + \beta_u^{\bar{v}}\left(\alpha_{\bar{v},r\bar{l}} - \alpha_{\bar{l},r\bar{v}}\right) \right] (\alpha_{k,\bar{q}p} - \alpha_{p,\bar{q}k}),
\end{aligned}
$$

as required. $\square$

Lemma 5.9. *Given the setup above,*

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_{g_\alpha}\right)\left(\partial\alpha_{ar}g^{\bar{s}r}\overline{\partial\alpha}_{\bar{s}\bar{b}}\right)\\
=&\ -\partial\alpha_{ar}\overline{\partial\alpha}_{\bar{s}\bar{b}}g^{\bar{s}i}g^{\bar{j}r}g^{\bar{l}p}g^{\bar{q}k}\left[\left(\alpha_{\bar{q},p\bar{j}}-\alpha_{\bar{j},p\bar{q}}\right)\left(\alpha_{k,\bar{l}i}-\alpha_{i,\bar{l}k}\right)+\left(\alpha_{i,\bar{q}l}-\alpha_{\bar{q},il}\right)\left(\alpha_{\bar{j},kp}-\alpha_{k,\bar{j}p}\right)\right]\\
&-g^{\bar{l}k}g^{\bar{s}r}\left[\left(\alpha_{a,rk}-\alpha_{r,ak}\right)\left(\alpha_{\bar{s},\bar{b}\bar{l}}-\alpha_{\bar{b},\bar{s}\bar{l}}\right)+\left(\alpha_{a,r\bar{l}}-\alpha_{r,a\bar{l}}\right)\left(\alpha_{\bar{s},\bar{b}k}-\alpha_{\bar{b},\bar{s}k}\right)\right]\\
&-\sqrt{-1}\overline{\partial\alpha}_{\bar{s}\bar{b}}g^{\bar{l}k}g^{\bar{s}p}g^{\bar{q}r}\Big[\alpha_{k,\bar{q}p}\alpha_{a,\bar{l}r}-\alpha_{k,\bar{q}p}\alpha_{\bar{l},ar}+\alpha_{\bar{q},ka}\alpha_{p,\bar{l}r}-\alpha_{a,rk}\alpha_{p,\bar{q}\bar{l}}\\
&\qquad\qquad+\alpha_{a,rk}\alpha_{\bar{q},p\bar{l}}+\alpha_{r,ak}\alpha_{p,\bar{q}\bar{l}}-\alpha_{r,ak}\alpha_{\bar{q},p\bar{l}}-\alpha_{a,r\bar{l}}\alpha_{p,\bar{q}k}\Big]\\
&+\sqrt{-1}\partial\alpha_{ar}g^{\bar{l}k}g^{\bar{s}p}g^{\bar{q}r}\Big[\alpha_{p,\bar{q}k}\alpha_{\bar{s},\bar{b}\bar{l}}-\alpha_{p,\bar{q}k}\alpha_{\bar{b},\bar{s}\bar{l}}-\alpha_{\bar{q},pk}\alpha_{\bar{s},\bar{b}\bar{l}}+\alpha_{\bar{q},pk}\alpha_{\bar{b},\bar{s}\bar{l}}\\
&\qquad\qquad+\alpha_{\bar{q},p\bar{l}}\alpha_{\bar{b},\bar{s}k}-\alpha_{\bar{l},p\bar{q}}\alpha_{\bar{b},\bar{s}k}+\alpha_{\bar{l},p\bar{q}}\alpha_{k,\bar{b}\bar{s}}-\alpha_{p,\bar{l}\bar{b}}\alpha_{\bar{q},\bar{s}k}\Big].
\end{aligned}
$$

Proof. We first of all compute

$$
\begin{aligned}
&\Delta_{g_\alpha}\left(\partial\alpha_{ar}g^{\bar{s}r}\overline{\partial\alpha}_{\bar{s}\bar{b}}\right)\\
&\quad=g^{\bar{l}k}\left[\left(\partial\alpha_{ar}g^{\bar{s}r}\right)\overline{\partial\alpha}_{\bar{s}\bar{b}}\right]_{,k\bar{l}}\\
&\quad=\Delta_{g_\alpha}\left[\partial\alpha_{ar}g^{\bar{s}r}\right]\overline{\partial\alpha}_{\bar{s}b}+\left(\partial\alpha_{ar}g^{\bar{s}r}\right)\Delta_{g_\alpha}\overline{\partial\alpha}_{\bar{s}\bar{b}}+g^{\bar{l}k}\left(\partial\alpha_{ar}g^{\bar{s}r}\right)_{,k}(\overline{\partial\alpha})_{\bar{s}\bar{b},\bar{l}}+g^{\bar{l}k}\left(\partial\alpha_{ar}g^{\bar{s}r}\right)_{,\bar{l}}(\overline{\partial\alpha})_{\bar{s}\bar{b},k}\\
&\quad=\Delta_{g_\alpha}\left[\partial\alpha_{ar}g^{\bar{s}r}\right]\overline{\partial\alpha}_{\bar{s}b}+\left(\partial\alpha_{ar}g^{\bar{s}r}\right)\Delta_{g_\alpha}\overline{\partial\alpha}_{\bar{s}\bar{b}}\\
&\qquad+g^{\bar{l}k}g^{\bar{s}r}\left[\left(\alpha_{a,rk}-\alpha_{r,ak}\right)\left(\alpha_{\bar{s},\bar{b}\bar{l}}-\alpha_{\bar{b},\bar{s}\bar{l}}\right)+\left(\alpha_{a,r\bar{l}}-\alpha_{r,a\bar{l}}\right)\left(\alpha_{\bar{s},\bar{b}k}-\alpha_{\bar{b},\bar{s}k}\right)\right]\\
&\qquad-\partial\alpha_{ar}g^{\bar{l}k}g^{\bar{s}p}g_{p\bar{q},k}g^{\bar{q}r}\left(\alpha_{\bar{s},\bar{b}\bar{l}}-\alpha_{\bar{b},\bar{s}\bar{l}}\right)-\partial\alpha_{ar}g^{\bar{l}k}g^{\bar{s}p}g_{p\bar{q},\bar{l}}g^{\bar{q}r}\left(\alpha_{\bar{s},\bar{b}k}-\alpha_{\bar{b},\bar{s}k}\right)\\
&\quad=\Delta_{g_\alpha}\left[\partial\alpha_{ar}g^{\bar{s}r}\right]\overline{\partial\alpha}_{\bar{s}b}+\left(\partial\alpha_{ar}g^{\bar{s}r}\right)\Delta_{g_\alpha}\overline{\partial\alpha}_{\bar{s}\bar{b}}\\
&\qquad+g^{\bar{l}k}g^{\bar{s}r}\left[\left(\alpha_{a,rk}-\alpha_{r,ak}\right)\left(\alpha_{\bar{s},\bar{b}\bar{l}}-\alpha_{\bar{b},\bar{s}\bar{l}}\right)+\left(\alpha_{a,r\bar{l}}-\alpha_{r,a\bar{l}}\right)\left(\alpha_{\bar{s},\bar{b}k}-\alpha_{\bar{b},\bar{s}k}\right)\right]\\
&\qquad-\sqrt{-1}\partial\alpha_{ar}g^{\bar{l}k}g^{\bar{s}p}g^{\bar{q}r}\left(\alpha_{p,\bar{q}k}-\alpha_{\bar{q},pk}\right)\left(\alpha_{\bar{s},\bar{b}\bar{l}}-\alpha_{\bar{b},\bar{s}\bar{l}}\right)\\
&\qquad-\sqrt{-1}\partial\alpha_{ar}g^{\bar{l}k}g^{\bar{s}p}g^{\bar{q}r}\left(\alpha_{p,\bar{q}\bar{l}}-\alpha_{\bar{q},p\bar{l}}\right)\left(\alpha_{\bar{s},\bar{b}k}-\alpha_{\bar{b},\bar{s}k}\right).
\end{aligned}
$$

Also we have the basic calculation

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta_{g_\alpha}\right)\left(\partial\alpha_{ar}g^{\bar{s}r}\overline{\partial\alpha}_{\bar{s}\bar{b}}\right)\\
=&\left[\frac{\partial}{\partial t}\left(\partial\alpha_{ar}g^{\bar{s}r}\right)\right]\overline{\partial\alpha}_{\bar{s}\bar{b}}+\left(\partial\alpha_{ar}g^{\bar{s}r}\right)\frac{\partial}{\partial t}\overline{\partial\alpha}_{\bar{s}\bar{b}}-\Delta_{g_\alpha}\left(\partial\alpha_{ar}g^{\bar{s}r}\overline{\partial\alpha}_{\bar{s}\bar{b}}\right)\\
=&\left[\left(\frac{\partial}{\partial t}-\Delta_{g_\alpha}\right)\left(\sqrt{-1}\partial\alpha_{ar}g^{\bar{s}r}\right)\right]\left(-\sqrt{-1}\overline{\partial\alpha}_{\bar{s}\bar{b}}\right)+\left(\sqrt{-1}\partial\alpha_{ar}g^{\bar{s}r}\right)\left(\frac{\partial}{\partial t}-\Delta_{g_\alpha}\right)\left(-\sqrt{-1}\overline{\partial\alpha}_{\bar{s}\bar{b}}\right)\\
&+\left[\left(\Delta_{g_\alpha}\partial\alpha_{ar}g^{\bar{s}r}\right)\overline{\partial\alpha}_{\bar{s}\bar{b}}+\left(\partial\alpha_{ar}g^{\bar{s}r}\right)\Delta_{g_\alpha}\overline{\partial\alpha}_{\bar{s}\bar{b}}-\Delta_{g_\alpha}\left(\partial\alpha_{ar}g^{\bar{s}r}\overline{\partial\alpha}_{\bar{s}\bar{b}}\right)\right]\\
=&\ E_1+E_2+E_3.
\end{aligned}
$$

Using Lemma 5.8 we compute

$$\begin{aligned}
E_1 = & -\partial\alpha_{ar}\overline{\partial\alpha}_{\overline{s}\overline{b}} g^{\overline{s}i} g^{\overline{j}r} g^{\overline{l}p} g^{\overline{q}k}\left[\left(\alpha_{\overline{q},p\overline{j}} - \alpha_{\overline{j},p\overline{q}}\right)\left(\alpha_{k,\overline{l}i} - \alpha_{i,\overline{l}k}\right) + \left(\alpha_{i,\overline{ql}} - \alpha_{\overline{q},i\overline{l}}\right)\left(\alpha_{\overline{j},kp} - \alpha_{k,\overline{j}p}\right)\right]\\
& - \sqrt{-1}\overline{\partial\alpha}_{\overline{s}\overline{b}} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}\Big[\alpha_{k,\overline{q}p}\alpha_{a,\overline{l}r} - \alpha_{k,\overline{q}p}\alpha_{\overline{l},ar} + \alpha_{\overline{q},ka}\alpha_{p,\overline{l}r} - \alpha_{a,rk}\alpha_{p,\overline{ql}}\\
& \qquad + \alpha_{a,rk}\alpha_{\overline{q},p\overline{l}} + \alpha_{r,ak}\alpha_{p,\overline{ql}} - \alpha_{r,ak}\alpha_{\overline{q},p\overline{l}} - \alpha_{a,r\overline{l}}\alpha_{p,\overline{q}k}\Big].
\end{aligned}$$

Also using Lemma 5.7 we have

$$E_2 = \sqrt{-1}\partial\alpha_{ar} g^{\overline{s}r} g^{\overline{l}p} g^{\overline{q}k}\left[\alpha_{p,\overline{qs}}\alpha_{\overline{b},\overline{l}k} - \alpha_{p,\overline{qs}}\alpha_{\overline{l},k\overline{b}} - \alpha_{\overline{q},p\overline{s}}\alpha_{\overline{b},\overline{l}k} + \alpha_{\overline{q},p\overline{s}}\alpha_{k,\overline{bl}} - \alpha_{p,\overline{qb}}\alpha_{\overline{s},\overline{l}k} + \alpha_{\overline{q},p\overline{b}}\alpha_{\overline{s},\overline{l}k}\right].$$

Also from the calculation above we have

$$\begin{aligned}
E_3 = & -g^{\overline{l}k} g^{\overline{s}r}\left[\left(\alpha_{a,rk} - \alpha_{r,ak}\right)\left(\alpha_{\overline{s},\overline{bl}} - \alpha_{\overline{b},\overline{sl}}\right) + \left(\alpha_{a,r\overline{l}} - \alpha_{r,a\overline{l}}\right)\left(\alpha_{\overline{s},\overline{b}k} - \alpha_{\overline{b},\overline{s}k}\right)\right]\\
& + \sqrt{-1}\partial\alpha_{ar} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}\left(\alpha_{p,\overline{q}k} - \alpha_{\overline{q},pk}\right)\left(\alpha_{\overline{s},\overline{bl}} - \alpha_{\overline{b},\overline{sl}}\right)\\
& + \sqrt{-1}\partial\alpha_{ar} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}\left(\alpha_{p,\overline{ql}} - \alpha_{\overline{q},p\overline{l}}\right)\left(\alpha_{\overline{s},\overline{b}k} - \alpha_{\overline{b},\overline{s}k}\right)\\
= & -g^{\overline{l}k} g^{\overline{s}r}\left[\left(\alpha_{a,rk} - \alpha_{r,ak}\right)\left(\alpha_{\overline{s},\overline{bl}} - \alpha_{\overline{b},\overline{sl}}\right) + \left(\alpha_{a,r\overline{l}} - \alpha_{r,a\overline{l}}\right)\left(\alpha_{\overline{s},\overline{b}k} - \alpha_{\overline{b},\overline{s}k}\right)\right]\\
& + \sqrt{-1}\partial\alpha_{ar} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}\Big[\alpha_{p,\overline{q}k}\alpha_{\overline{s},\overline{bl}} - \alpha_{p,\overline{q}k}\alpha_{\overline{b},\overline{sl}} - \alpha_{\overline{q},pk}\alpha_{\overline{s},\overline{bl}} + \alpha_{\overline{q},pk}\alpha_{\overline{b},\overline{sl}}\\
& \qquad + \alpha_{p,\overline{ql}}\alpha_{\overline{s},\overline{b}k} - \alpha_{p,\overline{ql}}\alpha_{\overline{b},\overline{s}k} - \alpha_{\overline{q},p\overline{l}}\alpha_{\overline{s},\overline{b}k} + \alpha_{\overline{q},p\overline{l}}\alpha_{\overline{b},\overline{s}k}\Big].
\end{aligned}$$

Collecting these calculations and relabeling indices yields

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right)\left(\partial\alpha_{ar} g^{\overline{s}r}\overline{\partial\alpha}_{\overline{s}\overline{b}}\right)\\
= & -\partial\alpha_{ar}\overline{\partial\alpha}_{\overline{s}\overline{b}} g^{\overline{s}i} g^{\overline{j}r} g^{\overline{l}p} g^{\overline{q}k}\left[\left(\alpha_{\overline{q},p\overline{j}} - \alpha_{\overline{j},p\overline{q}}\right)\left(\alpha_{k,\overline{l}i} - \alpha_{i,\overline{l}k}\right) + \left(\alpha_{i,\overline{ql}} - \alpha_{\overline{q},i\overline{l}}\right)\left(\alpha_{\overline{j},kp} - \alpha_{k,\overline{j}p}\right)\right]\\
& - g^{\overline{l}k} g^{\overline{s}r}\left[\left(\alpha_{a,rk} - \alpha_{r,ak}\right)\left(\alpha_{\overline{s},\overline{bl}} - \alpha_{\overline{b},\overline{sl}}\right) + \left(\alpha_{a,r\overline{l}} - \alpha_{r,a\overline{l}}\right)\left(\alpha_{\overline{s},\overline{b}k} - \alpha_{\overline{b},\overline{s}k}\right)\right]\\
& - \sqrt{-1}\overline{\partial\alpha}_{\overline{s}\overline{b}} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}\Big[\alpha_{k,\overline{q}p}\alpha_{a,\overline{l}r} - \alpha_{k,\overline{q}p}\alpha_{\overline{l},ar} + \alpha_{\overline{q},ka}\alpha_{p,\overline{l}r} - \alpha_{a,rk}\alpha_{p,\overline{ql}}\\
& \qquad + \alpha_{a,rk}\alpha_{\overline{q},p\overline{l}} + \alpha_{r,ak}\alpha_{p,\overline{ql}} - \alpha_{r,ak}\alpha_{\overline{q},p\overline{l}} - \alpha_{a,r\overline{l}}\alpha_{p,\overline{q}k}\Big]\\
& + \sqrt{-1}\partial\alpha_{ar} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}\Big[\alpha_{p,\overline{q}k}\alpha_{\overline{s},\overline{bl}} - \alpha_{p,\overline{q}k}\alpha_{\overline{b},\overline{sl}} - \alpha_{\overline{q},pk}\alpha_{\overline{s},\overline{bl}} + \alpha_{\overline{q},pk}\alpha_{\overline{b},\overline{sl}}\\
& \qquad + \alpha_{p,\overline{ql}}\alpha_{\overline{s},\overline{b}k} - \alpha_{p,\overline{ql}}\alpha_{\overline{b},\overline{s}k} - \alpha_{\overline{q},p\overline{l}}\alpha_{\overline{s},\overline{b}k} + \alpha_{\overline{q},p\overline{l}}\alpha_{\overline{b},\overline{s}k}\\
& \qquad + \alpha_{p,\overline{lq}}\alpha_{\overline{b},\overline{s}k} - \alpha_{p,\overline{lq}}\alpha_{\overline{s},k\overline{b}} - \alpha_{\overline{l},p\overline{q}}\alpha_{\overline{b},\overline{s}k} + \alpha_{\overline{l},p\overline{q}}\alpha_{k,\overline{bs}} - \alpha_{p,\overline{lb}}\alpha_{\overline{q},\overline{s}k} + \alpha_{\overline{l},p\overline{b}}\alpha_{\overline{q},\overline{s}k}\Big].
\end{aligned}$$

Let us label the final 14 terms above as $\sum_{i=1}^{14} A_i$ as usual. Then observe the cancellations $A_5 + A_{10} = A_6 + A_9 = A_7 + A_{14} = 0$. The result follows. □

Lemma 5.10. *Given the setup above,*

$$\begin{aligned}
& \left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right)\left[g_{a\overline{b}} - \partial\alpha_{ar} g^{\overline{s}r}\overline{\partial\alpha}_{\overline{s}\overline{b}}\right]\\
= & -g^{\overline{l}k} g^{\overline{s}r}\left[\left(\alpha_{a,r\overline{l}} - \alpha_{\overline{l},ar}\right) + \beta_a^{\overline{q}}\left(\alpha_{\overline{q},r\overline{l}} - \alpha_{\overline{l},r\overline{q}}\right)\right]\left[\left(\alpha_{\overline{b},\overline{s}k} - \alpha_{k,\overline{sb}}\right) + \beta_{\overline{b}}^{q}\left(\alpha_{q,\overline{s}k} - \alpha_{k,\overline{s}p}\right)\right]\\
& - g^{\overline{l}k} g^{\overline{s}r}\left[\left(\alpha_{a,rk} - \alpha_{r,ak}\right) + \beta_a^{\overline{q}}\left(\alpha_{\overline{q},rk} - \alpha_{r,\overline{q}k}\right)\right]\left[\left(\alpha_{\overline{b},\overline{sl}} - \alpha_{\overline{s},\overline{bl}}\right) + \beta_{\overline{b}}^{q}\left(\alpha_{q,\overline{sl}} - \alpha_{\overline{s},q\overline{l}}\right)\right].
\end{aligned}$$

Proof. We combine the results of Lemmas 5.5, 5.9. First we rewrite the $g^{-2}\partial^2\alpha^{*2}$ terms. These take the form

$$
\begin{aligned}
E = {}& g^{\bar l p} g^{\bar q k}\left[\alpha_{p,\bar q\bar b}\alpha_{a,\bar l k} - \alpha_{p,\bar q\bar b}\alpha_{\bar l,ak} - \alpha_{\bar q,p\bar b}\alpha_{a,\bar l k} + \alpha_{\bar q,p\bar b}\alpha_{k,\bar l a} - \alpha_{p,\bar q a}\alpha_{\bar b,k\bar l} + \alpha_{\bar q,pa}\alpha_{\bar b,k\bar l}\right]\\
&+ g^{\bar l k} g^{\bar s r}\left[(\alpha_{a,rk} - \alpha_{r,ak})\left(\alpha_{\bar s,\bar b l} - \alpha_{\bar b,\bar s l}\right) + \left(\alpha_{a,r\bar l} - \alpha_{r,a\bar l}\right)\left(\alpha_{\bar s,\bar b k} - \alpha_{\bar b,\bar s k}\right)\right]\\
= {}& g^{\bar l k} g^{\bar s r}\Big[(\alpha_{a,rk} - \alpha_{r,ak})\left(\alpha_{\bar s,\bar b l} - \alpha_{\bar b,\bar s l}\right) + \alpha_{a,r\bar l}\alpha_{\bar s,\bar b k} - \alpha_{a,r\bar l}\alpha_{\bar b,\bar s k} - \alpha_{r,a\bar l}\alpha_{\bar s,\bar b k} + \alpha_{r,a\bar l}\alpha_{\bar b,\bar s k}\\
&+\alpha_{k,\bar s\bar b}\alpha_{a,\bar l r} - \alpha_{k,\bar s\bar b}\alpha_{\bar l,ar} - \alpha_{\bar s,k\bar b}\alpha_{a,\bar l r} + \alpha_{\bar s,k\bar b}\alpha_{r,\bar l a} - \alpha_{k,\bar s a}\alpha_{\bar b,r\bar l} + \alpha_{\bar s,ka}\alpha_{\bar b,r\bar l}\Big]\\
= {}& g^{\bar l k} g^{\bar s r}\left[(\alpha_{a,rk} - \alpha_{r,ak})\left(\alpha_{\bar s,\bar b l} - \alpha_{\bar b,\bar s l}\right) - \alpha_{a,r\bar l}\alpha_{\bar b,\bar s k} + \alpha_{k,\bar s\bar b}\alpha_{a,\bar l r} - \alpha_{k,\bar s\bar b}\alpha_{\bar l,ar} + \alpha_{\bar s,ka}\alpha_{\bar b,r\bar l}\right]\\
= {}& g^{\bar l k} g^{\bar s r}\left[(\alpha_{a,rk} - \alpha_{r,ak})\left(\alpha_{\bar s,\bar b l} - \alpha_{\bar b,\bar s l}\right) - \left(\alpha_{a,r\bar l} - \alpha_{\bar l,ar}\right)\left(\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s\bar b}\right)\right]\\
= {}& -g^{\bar l k} g^{\bar s r}\left[(\alpha_{a,rk} - \alpha_{r,ak})\left(\alpha_{\bar b,\bar s l} - \alpha_{\bar s,\bar b l}\right) + \left(\alpha_{a,r\bar l} - \alpha_{\bar l,ar}\right)\left(\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s\bar b}\right)\right].
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right)\left[g_{a\bar b} - \partial\alpha_{ar} g^{\bar s r}\overline{\partial\alpha}_{\bar s\bar b}\right]\\
= {}& -\partial\alpha_{ar}\overline{\partial\alpha}_{\bar b\bar s} g^{\bar s i} g^{\bar j r} g^{\bar l p} g^{\bar q k}\left[\left(\alpha_{\bar q,p\bar j} - \alpha_{\bar j,p\bar q}\right)\left(\alpha_{k,\bar l i} - \alpha_{i,\bar l k}\right) + \left(\alpha_{i,\bar q\bar l} - \alpha_{\bar q,i\bar l}\right)\left(\alpha_{\bar j,kp} - \alpha_{k,\bar j p}\right)\right]\\
&- g^{\bar l k} g^{\bar s r}\left[(\alpha_{a,rk} - \alpha_{r,ak})\left(\alpha_{\bar b,\bar s l} - \alpha_{\bar s,\bar b l}\right) + \left(\alpha_{a,r\bar l} - \alpha_{\bar l,ar}\right)\left(\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s\bar b}\right)\right]\\
&+ \sqrt{-1}\,\overline{\partial\alpha}_{\bar s\bar b} g^{\bar l k} g^{\bar s p} g^{\bar q r}\Big[\alpha_{k,\bar q p}\alpha_{a,\bar l r} - \alpha_{k,\bar q p}\alpha_{\bar l,ar} + \alpha_{\bar q,ka}\alpha_{p,\bar l r} - \alpha_{a,rk}\alpha_{p,\bar q\bar l}\\
&\qquad +\alpha_{a,rk}\alpha_{\bar q,p\bar l} + \alpha_{r,ak}\alpha_{p,\bar q\bar l} - \alpha_{r,ak}\alpha_{\bar q,p\bar l} - \alpha_{a,r\bar l}\alpha_{p,\bar q k}\Big]\\
&- \sqrt{-1}\,\partial\alpha_{ar} g^{\bar l k} g^{\bar s p} g^{\bar q r}\Big[\alpha_{p,\bar q k}\alpha_{\bar s,\bar b l} - \alpha_{p,\bar q k}\alpha_{\bar b,\bar s l} - \alpha_{\bar q,pk}\alpha_{\bar s,\bar b l} + \alpha_{\bar q,pk}\alpha_{\bar b,\bar s l}\\
&\qquad +\alpha_{\bar q,p\bar l}\alpha_{\bar b,\bar s k} - \alpha_{\bar l,p\bar q}\alpha_{\bar b,\bar s k} + \alpha_{\bar l,p\bar q}\alpha_{k,\bar b\bar s} - \alpha_{p,\bar l\bar b}\alpha_{\bar q,\bar s k}\Big]\\
= {}& -\partial\alpha_{ar}\overline{\partial\alpha}_{\bar b\bar s} g^{\bar s i} g^{\bar j r} g^{\bar l p} g^{\bar q k}\left[\left(\alpha_{\bar q,p\bar j} - \alpha_{\bar j,p\bar q}\right)\left(\alpha_{k,\bar l i} - \alpha_{i,\bar l k}\right) + \left(\alpha_{i,\bar q\bar l} - \alpha_{\bar q,i\bar l}\right)\left(\alpha_{\bar j,kp} - \alpha_{k,\bar j p}\right)\right]\\
&- g^{\bar l k} g^{\bar s r}\left[(\alpha_{a,rk} - \alpha_{r,ak})\left(\alpha_{\bar b,\bar s l} - \alpha_{\bar s,\bar b l}\right) + \left(\alpha_{a,r\bar l} - \alpha_{\bar l,ar}\right)\left(\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s\bar b}\right)\right]\\
&+ \sqrt{-1}\,\overline{\partial\alpha}_{\bar s\bar b} g^{\bar l k} g^{\bar s p} g^{\bar q r}\left[(\alpha_{k,\bar q p} - \alpha_{p,\bar q k})\left(\alpha_{a,\bar l r} - \alpha_{\bar l,ar}\right) + (\alpha_{a,rk} - \alpha_{r,ak})(\alpha_{\bar q,p\bar l} - \alpha_{p,\bar q\bar l})\right]\\
&- \sqrt{-1}\,\partial\alpha_{ar} g^{\bar l k} g^{\bar s p} g^{\bar q r}\left[(\alpha_{p,\bar q k} - \alpha_{\bar q,pk})(\alpha_{\bar s,\bar b l} - \alpha_{\bar b,\bar s l}) + (\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s\bar b})(\alpha_{\bar q,p\bar l} - \alpha_{\bar l,p\bar q})\right]\\
= {}& \sum_{i=1}^{8} A_i.
\end{aligned}
$$

We combine terms, relabeling indices to yield

$$
\begin{aligned}
&A_1 + A_4 + A_5 + A_8\\
= {}& -\beta_{ar}\overline{\beta}_{\bar b\bar s} g^{\bar s i} g^{\bar j r} g^{\bar l p} g^{\bar q k}\left(\alpha_{\bar q,p\bar j} - \alpha_{\bar j,p\bar q}\right)\left(\alpha_{k,\bar l i} - \alpha_{i,\bar l k}\right) - g^{\bar l k} g^{\bar s r}\left(\alpha_{a,r\bar l} - \alpha_{\bar l,ar}\right)\left(\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s b}\right)\\
&+ \beta_{\bar b\bar s} g^{\bar l k} g^{\bar s p} g^{\bar q r}\left(\alpha_{k,\bar q p} - \alpha_{p,\bar q k}\right)\left(\alpha_{a,\bar l r} - \alpha_{\bar l,ar}\right) - \beta_{ar} g^{\bar l k} g^{\bar s p} g^{\bar q r}(\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s\bar b})(\alpha_{\bar q,p\bar l} - \alpha_{\bar l,p\bar q})\\
= {}& -g^{\bar l k} g^{\bar s r}\left[\left(\alpha_{a,r\bar l} - \alpha_{\bar l,ar}\right) + \beta_a^{\bar q}\left(\alpha_{\bar q,r\bar l} - \alpha_{\bar l,r\bar q}\right)\right]\left[\left(\alpha_{\bar b,\bar s k} - \alpha_{k,\bar s\bar b}\right) + \beta_{\bar b}^{q}\left(\alpha_{q,\bar s k} - \alpha_{k,\bar s p}\right)\right].
\end{aligned}
$$

Similarly we compute

$$\begin{aligned}
& A_2 + A_3 + A_6 + A_7 \\
= & -\beta_{ar}\beta_{\overline{b}\overline{s}} g^{\overline{s}i} g^{\overline{j}r} g^{\overline{l}p} g^{\overline{q}k} \left(\alpha_{i,\overline{q}\overline{l}} - \alpha_{\overline{q},i\overline{l}}\right)\left(\alpha_{\overline{j},kp} - \alpha_{k,\overline{j}p}\right) - g^{\overline{l}k} g^{\overline{s}r}\left(\alpha_{a,rk} - \alpha_{r,ak}\right)\left(\alpha_{\overline{b},\overline{s}\overline{l}} - \alpha_{\overline{s},\overline{b}\overline{l}}\right) \\
& + \beta_{\overline{b}\overline{s}} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}(\alpha_{a,rk} - \alpha_{r,ak})(\alpha_{\overline{q},p\overline{l}} - \alpha_{p,\overline{q}\overline{l}}) - \beta_{ar} g^{\overline{l}k} g^{\overline{s}p} g^{\overline{q}r}(\alpha_{p,\overline{q}k} - \alpha_{\overline{q},pk})(\alpha_{\overline{s},\overline{b}\overline{l}} - \alpha_{\overline{b},\overline{s}\overline{l}}) \\
= & -g^{\overline{l}k} g^{\overline{s}r}\left[(\alpha_{a,rk} - \alpha_{r,ak}) + \beta_a^{\overline{q}}(\alpha_{\overline{q},rk} - \alpha_{r,\overline{q}k})\right]\left[\left(\alpha_{\overline{b},\overline{s}\overline{l}} - \alpha_{\overline{s},\overline{b}\overline{l}}\right) + \beta_{\overline{b}}^{q}\left(\alpha_{q,\overline{s}\overline{l}} - \alpha_{\overline{s},q\overline{l}}\right)\right].
\end{aligned}$$

The result follows. □

Proposition 5.11. *Let α_t be a solution of (5.2) such that $\sqrt{-1}(\overline{\partial}\alpha + \overline{\partial\alpha}) > 0$ for all t. Then*

$$\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) W(\alpha_t) \leq 0.$$

Proof. As the calculations above are in an arbitrary coordinate basis, given a vector $v = (v_1, v_2) \in T^{1,0}\mathbb{C}^{2n}$ we can extend it to a coordinate basis and apply Lemmas 5.6, 5.8 and 5.10 to yield

$$\begin{aligned}
& \left[\left(\frac{\partial}{\partial t} - \Delta_{g_\alpha}\right) W\right](v, \overline{v}) \\
= & -g^{\overline{v}_2 i} g^{\overline{j} v_2} g^{\overline{l}p} g^{\overline{q}k}\left[\left(\alpha_{\overline{q},p\overline{j}} - \alpha_{\overline{j},p\overline{q}}\right)\left(\alpha_{k,\overline{l}i} - \alpha_{i,\overline{l}k}\right) + \left(\alpha_{i,\overline{q}\overline{l}} - \alpha_{\overline{q},i\overline{l}}\right)\left(\alpha_{\overline{j},kp} - \alpha_{k,\overline{j}p}\right)\right] \\
& - g^{\overline{l}k} g^{\overline{s}r}\left[\left(\alpha_{v_1,r\overline{l}} - \alpha_{\overline{l},v_1 r}\right) + \beta_{v_1}^{\overline{q}}\left(\alpha_{\overline{q},r\overline{l}} - \alpha_{\overline{l},r\overline{q}}\right)\right]\left[(\alpha_{\overline{v}_1,\overline{s}k} - \alpha_{k,\overline{s}\overline{v}_1}) + \beta_{\overline{v}_1}^{q}(\alpha_{q,\overline{s}k} - \alpha_{k,\overline{s}p})\right] \\
& - g^{\overline{l}k} g^{\overline{s}r}\left[(\alpha_{v_1,rk} - \alpha_{r,v_1 k}) + \beta_{v_1}^{\overline{q}}(\alpha_{\overline{q},rk} - \alpha_{r,\overline{q}k})\right]\left[\left(\alpha_{\overline{v}_1,\overline{s}\overline{l}} - \alpha_{\overline{s},\overline{v}_1\overline{l}}\right) + \beta_{\overline{v}_1}^{q}\left(\alpha_{q,\overline{s}\overline{l}} - \alpha_{\overline{s},q\overline{l}}\right)\right] \\
& - g^{\overline{l}k} g^{\overline{v}_2 p} g^{\overline{q}r}\left[(\alpha_{v_1,rk} - \alpha_{r,v_1 k}) + \beta_{v_1}^{\overline{s}}(\alpha_{\overline{s},rk} - \alpha_{r,\overline{s}k})\right]\left(\alpha_{p,\overline{q}\overline{l}} - \alpha_{\overline{q},p\overline{l}}\right) + \text{conj.} \\
& + g^{\overline{l}k} g^{\overline{v}_2 p} g^{\overline{q}r}\left[\left(\alpha_{v_1,\overline{l}r} - \alpha_{\overline{l},v_1 r}\right) + \beta_{v_1}^{\overline{s}}\left(\alpha_{\overline{s},r\overline{l}} - \alpha_{\overline{l},r\overline{s}}\right)\right](\alpha_{k,\overline{q}p} - \alpha_{p,\overline{q}k}) + \text{conj.} \\
= & \sum_{i=1}^{8} A_i,
\end{aligned}$$

where we have labeled each term, including the conjugate terms. Using the Cauchy-Schwarz inequality we conclude

$$A_5 + A_6 \leq A_4 + A_2, \qquad A_7 + A_8 \leq A_1 + A_3.$$

□

We now rewrite these estimates in a different framework. In particular, we note that the matrix W only involves the “gauge-invariant” quantities g and β, subject to the integrability condition $\partial\omega = \overline{\partial}\beta$, or, in coordinates,

$$\beta_{ij,\overline{k}} = g_{i\overline{k},j} - g_{j\overline{k},i} \tag{5.3}$$

To that end the entire discussion can be expressed using these quantities, as we now observe. In particular, one can reinterpret the evolution equations above as the system

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_g\right) g_{i\overline{j}} = & -g^{\overline{m}p} g^{\overline{q}n}\left[g_{n\overline{m},i} g_{p\overline{j},\overline{q}} + g_{n\overline{m},\overline{j}} g_{i\overline{q},p} - g_{p\overline{q},i} g_{n\overline{m},\overline{j}}\right], \\
\left(\frac{\partial}{\partial t} - \Delta_g\right) \beta_{ij} = & -g^{\overline{q}p} g^{\overline{s}r}\left[g_{i\overline{s},p}\beta_{rj,\overline{q}} + g_{j\overline{s},p}\beta_{ir,\overline{q}}\right].
\end{aligned} \tag{5.4}$$

Lemma 5.12. *Let (g_t, β_t) be a solution to (5.4) satisfying (5.3). Then*

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} - \Delta_g\right) \left[g_{a\bar{b}} - \beta_{ar} g^{\bar{s}r} \beta_{\bar{s}\bar{b}}\right] \\
&\qquad = - g^{\bar{l}k} g^{\bar{s}r} \left[\left(g_{a\bar{l},r} + \beta_a^{\overline{q}} \beta_{\overline{q}\bar{l},r} \right) \left(g_{\bar{b}k,\bar{s}} + \beta_{\bar{b}}^{q} \beta_{qk,\bar{s}} \right) + \left(\beta_{ar,k} + \beta_a^{\overline{q}} g_{\overline{q}r,k} \right) \left(\beta_{\overline{b}\overline{s},\bar{l}} + \beta_{\bar{b}}^{q} g_{q\bar{s},\bar{l}} \right) \right],
\end{aligned}$$

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) g^{\bar{s}r} = - g^{\bar{s}i} g^{\bar{j}r} g^{\bar{l}p} g^{\bar{q}k} \left(\beta_{\overline{q}\overline{j},p} \beta_{ki,\bar{l}} + g_{i\overline{q},\bar{l}} g_{\bar{j}k,p} \right),$$

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) \beta_u^{\bar{s}} = - g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[\beta_{ur,k} + \beta_u^{\overline{v}} g_{\overline{v}r,k} \right] g_{p\overline{q},\bar{l}} + g^{\bar{l}k} g^{\bar{s}p} g^{\bar{q}r} \left[g_{u\bar{l},r} + \beta_u^{\overline{v}} \beta_{\overline{v}\bar{l},r} \right] \beta_{kp,\overline{q}}.$$

Proposition 5.13. *Let (g_t, β_t) be a solution to (5.4) satisfying (5.3). Let $W(g, \beta)$ be given by Definition 5.2. Then*

$$\left(\frac{\partial}{\partial t} - \Delta_g\right) W(g, \beta) \leq 0.$$

5.3. **Proof of Theorem 1.7.** In this subsection we establish the Evans-Krylov estimate for the pluriclosed flow. The structure of the proof exploits ideas similar to those of ([40] Theorem 1.1), which themselves are closely modeled after the proof of Evans-Krylov. The main step is to establish C^α regularity of W for a uniformly parabolic solution to (5.2), after which Theorem 1.7 follows from Schauder estimates and blowup arguments. The proof is closely modeled after ([28] Lemma 14.6), relying crucially on Lemma 5.4 and Proposition 5.11. To begin we recall some standard notation and results.

Definition 5.14. Given $(w, s) \in \mathbb{C}^n \times \mathbb{R}$, let

$$\begin{aligned}
Q((w, s), R) &:= \{(z, t) \in \mathbb{C}^n \times \mathbb{R} |\ t \leq s, \quad \max\{|z - w|, |t - s|^{\frac{1}{2}}\} < R\} \\
\Theta(R) &:= Q((w, s - 4R^2), R).
\end{aligned}$$

Theorem 5.15. *([28] Theorem 7.37) Let u be a nonnegative function on $Q(4R)$ such that*

$$-u_t + a^{ij} u_{ij} \leq 0,$$

where

$$\lambda \delta_i^j \leq a^{ij} \leq \Lambda \delta_i^j.$$

There are positive constants $C, p > 1$ depending only on n, λ, Λ such that

$$\left(R^{-n-2} \int_{\Theta(R)} u^p \right)^{\frac{1}{p}} \leq C \inf_{Q(R)} u. \tag{5.5}$$

Proposition 5.16. *Suppose $\alpha \in \mathcal{E}_{Q(R)}^{\lambda,\Lambda}$ satisfies (5.2). There are positive constants γ, C depending only on n, λ, Λ such that for all $\rho < R$,*

$$\operatorname{osc}_{Q(\rho)} W \leq C(n, \lambda, \Lambda) \left(\frac{\rho}{R}\right)^\gamma \operatorname{osc}_{Q(R)} W.$$

Proof. Note that the log det operator is (λ, Λ) elliptic on a convex set containing the range of W. Using this and Lemma 5.4 shows that for any two points $(x, t_1), (y, t_2) \in Q(4R)$ there exists a matrix a^{ij}, $\lambda \delta_i^j \leq a^{i\bar{j}} \leq \Lambda \delta_i^j$ such that

$$\begin{aligned}
0 =&\ \log \det W(x, t_1) - \log \det W(y, t_2) \\
=&\ a^{i\bar{j}}((x, t_1), (y, t_2)) \left(W_{i\bar{j}}(x, t_1) - W_{i\bar{j}}(y, t_2) \right).
\end{aligned} \tag{5.6}$$

It follows from ([28] Lemma 14.5) that we can choose unit vectors v_α and functions $f_\alpha = f_\alpha((x,t_1),(y,t_2))$, such that

$$a^{i\bar{j}} = \sum_{\alpha=1}^{N} f_\alpha v_\alpha^i \overline{v_\alpha^j},$$

and moreover $\lambda_* \leq f_\alpha \leq \Lambda_*$, where λ_*, Λ_* only depend on λ, Λ. Now let $w_\alpha := W_{v_\alpha \overline{v_\alpha}}$. Then (5.6) reads

$$\sum f_\alpha \left(w_\alpha(y,t_2) - w_\alpha(x,t_1) \right) = 0. \tag{5.7}$$

Let

$$M_{s\alpha} = \sup_{Q(sR)} w_\alpha, \qquad m_{s\alpha} = \inf_{Q(sR)} w_\alpha, \qquad P(sR) = \sum_\alpha M_{s\alpha} - m_{s\alpha}.$$

Observe using Proposition 5.11 that every $M_{2\alpha} - w_\alpha$ is a supersolution to a uniformly parabolic equation. Thus by Theorem 5.15 we conclude

$$\left(R^{-n-2} \int_{\Theta(R)} (M_{2\alpha} - w_\alpha)^p \right)^{\frac{1}{p}} \leq C \left(M_{2\alpha} - M_\alpha \right). \tag{5.8}$$

Now observe that (5.7) yields for every pair $(x,t_1),(y,t_2) \in Q(2R)$,

$$f_\alpha \left(w_\alpha(y,t_2) - w_\alpha(x,t_1) \right) = \sum_{\beta \neq \alpha} f_\beta \left(w_\beta(x,t_1) - w_\beta(y,t_2) \right).$$

It follows directly that

$$w_\alpha(y,t_2) - m_{2\alpha} \leq C \sum_{\beta \neq \alpha} M_{2\beta} - w_\beta(y,t_2).$$

Integrating this over $\Theta(R)$, applying Minkowski's inequality and (5.8) yields

$$\begin{aligned}
\left(R^{-n-2} \int_{\Theta(R)} (w_\alpha - m_{2\alpha})^p \right)^{\frac{1}{p}} \leq&\ \left(C R^{-n-2} \int_{\Theta(R)} \left(\sum_{\beta \neq \alpha} M_{2\beta} - w_\beta \right)^p \right)^{\frac{1}{p}} \\
\leq&\ C \sum_{\beta \neq \alpha} \left(R^{-n-2} \int_{\Theta(R)} (M_{2\beta} - w_\beta)^p \right)^{\frac{1}{p}} \\
\leq&\ C \sum_{\beta \neq \alpha} M_{2\beta} - M_\beta.
\end{aligned} \tag{5.9}$$

Now we use (5.8) and (5.9) and Minkowski's inequality to yield

$$\begin{aligned}
M_{2\beta} - m_{2\beta} =&\ \left(R^{-n-2} \int_{\Theta(R)} (M_{2\beta} - m_{2\beta})^p \right)^{\frac{1}{p}} \\
=&\ \left(R^{-n-2} \int_{\Theta(R)} (M_{2\beta} - w_\beta + (w_\beta - m_{2\beta}))^p \right)^{\frac{1}{p}} \\
\leq&\ C \sum_\alpha M_{2\alpha} - M_\alpha \\
\leq&\ C \sum_\alpha M_{2\alpha} - M_\alpha + m_\alpha - m_{2\alpha} \\
=&\ C \left(P(2R) - P(R) \right).
\end{aligned}$$

Summing over β and rearranging yields for some constant $0 < \mu < 1$ the inequality

$$P(R) \leq \mu P(2R).$$

A standard iteration argument now yields the statement of the theorem. $\square$

Proposition 5.16 is the crucial point in establishing Theorem 1.7. To use it we first obtain a C^α estimate on the metric, and then employ blowup arguments.

Corollary 5.17. *Suppose $\alpha \in \mathcal{E}_{Q(2)}^{\lambda,\Lambda}$ satisfies (5.2). There are positive constants γ, C depending only on n, λ, Λ such*

$$\left|g_\alpha\right|_{C^\gamma(Q(1))} \leq C.$$

Proof. A standard argument using Proposition 5.16 shows that for a solution α as in the statement, there are constants γ, C depending on n, λ, Λ such that $\left|W\right|_{C^\gamma(Q(1))} \leq C$. Examining the lower right block of W, this yields a C^γ estimate for g^{-1}. Using this and looking at the upper right and lower left blocks of W yields a C^γ estimate for $\partial\alpha$. Finally, combining these estimates and considering the upper left block of W yields a C^γ estimate for g, as required. $\square$

Next we bootstrap these estimates to get higher regularity for the metric. Since equation (5.2) is degenerate parabolic, we need to use the induced equation on the metric, which is strictly parabolic. To that end, for a Hermitian manifold (M^n, J, g), let h denote an auxiliary Hermitian metric, which in local calculations we will take to be a flat metric, and let $\Upsilon(g, h) = \nabla_g - \nabla_h$ be the difference of the two Chern connections. Furthermore, let

$$f_k = f_k(g, h) := \sum_{j=0}^{k} \left|\nabla_g^j \Upsilon(g, h)\right|^{\frac{2}{1+j}}.$$

We now state a basic smoothing estimate.

Lemma 5.18. *Fix constants λ, Λ, K, and let g_t be a solution to pluriclosed flow on $B_R(0) \times [0, T]$ such that*

$$\lambda g_E \leq g \leq \Lambda g_E, \qquad \sup_{B_R \times [0,T]} f_1(x, t) \leq K.$$

Given $k \in \mathbb{N}$, there exists $C = C(R, T, \lambda, \Lambda, K)$ such that

$$\sup_{B_{\frac{R}{2}} \times [\frac{T}{2}, T]} f_k(x, t) \leq C.$$

Proof. We recall that in complex coordinates the pluriclosed flow equation can be expressed as

$$\frac{\partial}{\partial t} g_{i\bar{j}} = g^{\bar{l}k} g_{i\bar{j},k\bar{l}} + \partial g * \partial g.$$

Since an estimate on f_1 implies a C^2 estimate for the metric itself, the lemma follows in a standard way using cutoff functions and Schauder estimates ([28] Theorem 4.9). $\square$

Proposition 5.19. *Suppose $\alpha \in \mathcal{E}_{Q(2)}^{\lambda,\Lambda}$ satisfies (5.2). Given $k \in \mathbb{N}$, there exists $C = C(n, \lambda, \Lambda, k)$ such that*

$$\sup_{Q(1)} f_k(g_\alpha) \leq C.$$

Proof. We use a blowup/contradiction argument. Fix constants $\lambda, \Lambda, k, \gamma$ as in the statement and suppose the statement were false. Choose a sequence of solutions $\{\alpha_t^i\}$ as in the statement satisfying

the hypotheses, but for which there exists a sequence $\lambda_i \to \infty$ together with points $(x_i, t_i) \in Q\,(1)$ such that

$$\lambda_i = f_k(g_i, x_i, t_i).$$

We now claim that for i sufficiently large there exists a new point $\widetilde{x}_i, \widetilde{t}_i$ such that

$$(\widetilde{x}_i, \widetilde{t}_i) \in Q\left(\frac{3}{2}\right), \qquad \widetilde{\lambda}_i := f_k(g_i, \widetilde{x}_i, \widetilde{t}_i) \geq \lambda_i, \qquad \sup_{Q(\widetilde{\lambda}_i^{-1}, \widetilde{x}_i, \widetilde{t}_i)} f_k \leq 2\widetilde{\lambda}_i. \tag{5.10}$$

To show this we make an inductive choice of points. Fix some i, and let $(y_0, s_0) = (x_i, t_i)$. By construction (y_0, s_0) satisfies the first two conditions of (5.10). Given now some point (y_j, s_j) satisfying the first two conditions of (5.10), if it satisfies the third condition we set $(\widetilde{x}_i, \widetilde{t}_i) = (y_j, s_j)$. Otherwise, set $\mu_j = f_k(g_i, y_j, s_j)$ and choose $(y_{j+1}, s_{j+1}) \in Q(\mu_j^{-1}, \widetilde{y}_j, \widetilde{s}_j)$ such that $f_k(g_i, y_{j+1}, s_{j+1}) \geq 2\mu_j$. Note that by construction, for any j we have

$$\left|s_N\right| \leq 1 + \sum_{j=1}^{N} \mu_j^{-1} \leq 1 + \lambda_i^{-1} \sum_{j=1}^{\infty} 2^{-j} < \frac{9}{4}.$$

for λ_i sufficiently large. A similar estimate shows that $\left|y_N\right| < \frac{3}{2}$ for any N. Thus our inductive choice is well-defined, and since the solution is smooth this process terminates at some finite N, finishing the proof of the claim.

We rescale around these points to finish the argument. In particular, let us simplify notation and consider triples (g_i, x_i, t_i) of solutions defined on $Q(2)$ with the points $(x_i, t_i) \in Q(\frac{3}{2})$, and $\lambda_i = f_k(g_i, x_i, t_i) \to \infty$, and finally

$$\sup_{Q(\lambda_i^{-1}, x_i, t_i)} f_k \leq 2\lambda_i.$$

Now let

$$\widetilde{g}_i(x, t) = g_i\left(x_i + \lambda_i^{-\frac{1}{2}} x, t_i + \lambda_i^{-1} t\right).$$

By construction each solution $\widetilde{g}_i$ is defined on $Q(1)$ with

$$\sup_{Q(1)} f_k(\widetilde{g}_i) = 1.$$

By Lemma 5.18, we conclude that there exists a subsequence of $\widetilde{g}_i$ converging in C^∞ to a limiting solution g_∞ such that $f_k(g_\infty, 0, 0) = 1$. But on the other hand by Corollary 5.17 we have an a priori C^α estimate for the metrics g_i at the points (x_i, t_i). After the blowup this implies that the metric g_∞ is constant in space and time, and so in particular $f_k(g_\infty, 0, 0) = 0$. This is a contradiction, finishing the proof. □

Corollary 5.20. *Let α_t be a solution to (5.2) on $(-\infty, 0] \times \mathbb{C}^n$ such that $\alpha_t \in \mathcal{E}_{\mathbb{C}^n, \lambda, \Lambda}$ for all $t \in (-\infty, 0]$. Then $\frac{\partial}{\partial x} g_\mu \equiv \frac{\partial}{\partial t} g_\mu \equiv 0$.*

Proof. Suppose there exists a point such that $\left|\frac{\partial}{\partial x} g_\alpha\right| \neq 0$. By translating in space and time we can assume without loss of generality this point is $(0, 0)$. Fix some $A > 0$ and consider

$$\mu(x, t) := A^{-1} \alpha(Ax, A^2 t).$$

By direct calculation one verifies that μ is a solution to (5.2) on $(-\infty, 0] \times \mathbb{C}^n$ and moreover $\mu \in \mathcal{E}_{\mathbb{C}^n}^{\lambda, \Lambda}$ for all $t \in (-\infty, 0]$. Also observe that $\left|\frac{\partial}{\partial x} g_\mu\right|(0, 0) = A \left|\frac{\partial}{\partial x} g_\alpha\right|(0, 0)$. For A chosen sufficiently large this contradicts the result of Proposition 5.19, finishing the proof. □

Proof of Theorem 1.7. If the statement of the theorem is false, then there exists a sequence $\{(g^i_t, \alpha^i_t, \hat{g}^i_t, h^i, \mu^i)\}$ of solutions such that $g_0, \hat{g}, h$ satisfy uniform geometric bounds (i.e. lower bounds on injectivity radii and uniform bounds on curvature and all covariant derivatives of curvature), but such that there exist points $(x_i, t_i) \in M_i \times [0, \tau)$, $\tau \leq 1$, such that

$$t_i f_k(g_{t_i}, h)(x_i) = \sup_{M_i \times [0,\tau)} t f_k(g_t, h) \to \infty.$$

We now perform a blowup, aiming to get a contradiction to Proposition 5.19. Fix a constant $A > 0$, let $\sigma_i := f_k(g_{t_i}, h)(x_i)$, and set

$$\lambda_i := (A\sigma_i)^{-\frac{1}{2}}.$$

Due to the uniform estimates on the background data, we can choose a small radius $R > 0$ so that around each point (x_i, t_i) we have a normal coordinate chart for $\hat{g}_i$ of radius R. Using these coordinates we define

$$\begin{aligned}
\alpha_i'(x,t) &= \lambda_i^{-1}\alpha_i(x_i + \lambda_i x, t_i + \lambda_i^2 t),\\
\mu_i'(x,t) &= \lambda_i^{-1}\mu_i(x_i + \lambda_i x, t_i + \lambda_i^2 t),\\
\hat{g}_i'(x,t) &= \hat{g}_i(x_i + \lambda_i x, t_i + \lambda_i^2 t),\\
h_i'(x,t) &= h_i(x_i + \lambda_i x, t_i + \lambda_i^2 t).
\end{aligned}$$

By simple computations one observes that the resulting blowup data still define a solution to reduced pluriclosed flow. We also observe by construction that for sufficiently large i each of the resulting blowup solutions exists on $[-1, 0] \times B_1(0)$. Also by construction, we obtain that for sufficiently large i,

$$f_k(g_i', h_i')(0,0) = A^{-1}, \qquad \sup_{[-1,0]\times B_1(0)} f_k(g', h) \leq 2A^{-1}. \tag{5.11}$$

We observe by construction that $\{\hat{g}_i'\}$ converges to the standard Euclidean metric on $B_1(0)$, constant in time. Moreover, the sequence $\{h_i'\}$ will converge to a different, time-independent flat metric on $B_1(0)$.

We note that by the metric estimates, one has that $\{\partial\alpha_i'\}$ has uniform C^k estimates along the sequence. We need to account for the gauge invariance to obtain a full estimate for α' however. We can choose a function $f_i \in C^\infty(M)$ such that $\partial^*_{\hat{g}_i'}[\alpha_i'(-1) + \partial f] = 0$. Moreover we can choose a constant $(1,0)$ form η_i such that $[\alpha_i'(-1) + \partial f_i + \eta_i](0, -1) = 0$. Let $\widetilde{\alpha}_i'(t) = \alpha_i'(t) + \partial f_i + \eta_i$. Using the evolution equation for α_i' and the estimates on f_k, one obtains a C^{k-2} estimate on $\partial^*_{\hat{g}_i'}\widetilde{\alpha}_i'(t)$ and a C^0 estimate on $\widetilde{\alpha}_i'$ on $B_1(0) \times [-1, 0]$. Using estimates coming from Hodge theory we obtain that $\{\widetilde{\alpha}_i'\}$ converges subsequentially to a solution to (5.4) on $[-1, 0] \times B_1(0)$ such that

$$f_k(g_\infty, h_\infty)(0,0) = A^{-1}.$$

For A chosen sufficiently small this contradicts Proposition 5.19. □

6. Global existence and convergence on nonpositively curved backgrounds

In this section we first prove Theorem 1.8, and then use it and Theorem 1.7 to establish Theorem 1.1.

6.1. **Proof of Theorem 1.8.** In this subsection we employ the evolution equations of §4 to establish Theorem 1.8. To begin we recall two evolution equations for pluriclosed flow from [40].

Lemma 6.1. *Let (M^{2n}, g_t, J) be a solution to pluriclosed flow, and let h denote another Hermitian metric on (M, J). Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \frac{\det g}{\det h} = \left|T\right|^2 - \operatorname{tr}_g \rho(h).$$

Proof. We directly compute using (2.1),

$$\begin{aligned}
\frac{\partial}{\partial t} \log \frac{\det g}{\det h} &= g^{\bar{j}i} \left(\frac{\partial}{\partial t} g\right)_{i\bar{j}} \\
&= g^{\bar{j}i} \left[g^{\bar{q}p} g_{i\bar{j},p\bar{q}} - g^{\bar{q}p} g^{\bar{s}r} g_{i\bar{s},p} g_{r\bar{j},\bar{q}} + Q_{i\bar{j}} \right].
\end{aligned}$$

Also

$$\begin{aligned}
\Delta \log \frac{\det g}{\det h} &= g^{\bar{q}p} \left[\log \frac{\det g}{\det h} \right]_{,p\bar{q}} \\
&= g^{\bar{q}p} \left[g^{\bar{j}i} g_{i\bar{j},p} - h^{\bar{j}i} h_{i\bar{j},p} \right]_{,\bar{q}} \\
&= g^{\bar{q}p} \left[g^{\bar{j}i} g_{i\bar{j},p\bar{q}} - g^{\bar{j}k} g_{k\bar{l},\bar{q}} g^{\bar{l}i} g_{i\bar{j},p} - h^{\bar{j}i} h_{i\bar{j},p\bar{q}} + h^{\bar{j}k} h_{k\bar{l},\bar{q}} h^{\bar{l}i} h_{i\bar{j},p} \right].
\end{aligned}$$

Combining the above calculations yields

$$\left(\frac{\partial}{\partial t} - \Delta\right) \log \frac{\det g}{\det h} = \left|T\right|^2 - \operatorname{tr}_g \rho(h),$$

as required. □

Lemma 6.2. *Let (M^{2n}, g_t, J) be a solution to pluriclosed flow, and let h denote another Hermitian metric on (M, J). Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right) \operatorname{tr}_g h = -\left|\Upsilon(g,h)\right|^2_{g^{-1},g^{-1},h} - \langle h, Q \rangle + \Omega_h(g,g).$$

Proof. We directly compute

$$\begin{aligned}
\frac{\partial}{\partial t} \operatorname{tr}_g h &= \frac{\partial}{\partial t} g^{\bar{j}i} h_{i\bar{j}} \\
&= -g^{\bar{j}k} \left(\frac{\partial}{\partial t} g_{k\bar{l}} \right) g^{\bar{l}i} h_{i\bar{j}} \\
&= -g^{\bar{j}k} g^{\bar{l}i} h_{i\bar{j}} \left[g^{\bar{q}p} g_{k\bar{l},p\bar{q}} - g^{\bar{q}p} g^{\bar{s}r} g_{k\bar{s},p} g_{r\bar{l},\bar{q}} + Q_{k\bar{l}} \right].
\end{aligned}$$

On the other hand

$$\begin{aligned}
\Delta \operatorname{tr}_g h &= g^{\bar{q}p} \left[g^{\bar{j}i} h_{i\bar{j}} \right]_{,p\bar{q}} \\
&= g^{\bar{q}p} \left[-g^{\bar{j}k} g_{k\bar{l},p} g^{\bar{l}i} h_{i\bar{j}} + g^{\bar{j}i} h_{i\bar{j},p} \right]_{,\bar{q}} \\
&= g^{\bar{q}p} \Big[g^{\bar{j}r} g_{r\bar{s},\bar{q}} g^{\bar{s}k} g_{k\bar{l},p} g^{\bar{l}i} h_{i\bar{j}} - g^{\bar{j}k} g_{k\bar{l},p\bar{q}} g^{\bar{l}i} h_{i\bar{j}} + g^{\bar{j}k} g_{k\bar{l},p} g^{\bar{l}r} g_{r\bar{s},\bar{q}} g^{\bar{s}i} h_{i\bar{j}} \\
&\qquad - g^{\bar{j}k} g_{k\bar{l},p} g^{\bar{l}i} h_{i\bar{j},\bar{q}} - g^{\bar{j}k} g_{k\bar{l},\bar{q}} g^{\bar{l}i} h_{i\bar{j},p} + g^{\bar{j}i} h_{i\bar{j},p\bar{q}} \Big].
\end{aligned}$$

Combining the above calculations yields

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\operatorname{tr}_g h =&\ -g^{\overline{q}p}g^{\overline{j}r}g^{\overline{s}k}g^{\overline{l}i}h_{i\overline{j}}g_{r\overline{s},\overline{q}}g_{k\overline{l},p}\\ &+g^{\overline{q}p}\left[g^{\overline{j}k}g_{k\overline{l},p}g^{\overline{l}i}h_{i\overline{j},\overline{q}}+g^{\overline{j}k}g_{k\overline{l},\overline{q}}g^{\overline{l}i}h_{i\overline{j},p}-g^{\overline{j}i}h_{i\overline{j},p\overline{q}}\right]-\langle h,Q\rangle_g\\ =&\ -\left|\Upsilon(g,h)\right|^2_{g^{-1},g^{-1},h}-\langle h,Q\rangle+\Omega_h(g,g).\end{aligned}$$

□

Proof of Theorem 1.8. Assuming the setup of the theorem, let

$$\Phi(x,t):=1+\operatorname{tr}_g h+\log\frac{\det g}{\det h}+2\left|\partial\alpha\right|^2.$$

Combining Proposition 4.9 and Lemmas 6.2, 6.1 yields

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\Delta\right)\Phi =&\ -\left|\Upsilon(g,h)\right|^2_{g^{-1},g^{-1},h}-\langle h,Q\rangle+\Omega_h(g,g)+|T|^2-\operatorname{tr}_g\rho(h)\\ &-2\left|\nabla\partial\alpha\right|^2-2\left|\overline{\nabla}\partial\alpha\right|^2-4\left\langle Q,\operatorname{tr}\partial\alpha\otimes\overline{\partial\alpha}\right\rangle+4\Re\sqrt{-1}\left\langle\operatorname{tr}_g\nabla T_{\hat{g}}+\partial\mu,\overline{\partial\alpha}\right\rangle.\end{aligned}$$

We proceed to estimate the various terms above. First using the lower bound of the metric,

$$\Omega_h(g,g)=g^{\overline{l}k}g^{\overline{j}i}(\Omega^h)_{i\overline{j}k\overline{l}}\leq C(\operatorname{tr}_g h)^2\leq C(\lambda,h,g_0).$$

Similarly

$$-\operatorname{tr}_g\rho(h)\leq C\operatorname{tr}_g h\leq C(\lambda,h,g_0).$$

Also we express

$$\begin{aligned}4\left\langle\operatorname{tr}_{g_\alpha}\nabla T_{\hat{g}},\overline{\partial\alpha}\right\rangle&=4g^{\overline{q}p}\nabla_p(T_{\hat{g}})_{ij\overline{q}}\overline{\partial\alpha}_{\overline{kl}}g^{\overline{k}i}g^{\overline{l}j}\\ &=4g^{\overline{q}p}g^{\overline{k}i}g^{\overline{l}j}\overline{\partial\alpha}_{\overline{kl}}\left[(\nabla^h)_p(T_{\hat{g}})_{ij\overline{q}}+\Upsilon(g,h)^r_{pi}(T_{\hat{g}})_{rj\overline{q}}+\Upsilon(g,h)^r_{pj}(T_{\hat{g}})_{ir\overline{q}}\right].\end{aligned}$$

Then we estimate in a basis where $h=\mathrm{Id}$ and g is diagonalized,

$$\begin{aligned}4g^{\overline{q}p}g^{\overline{k}i}g^{\overline{l}j}\overline{\partial\alpha}_{\overline{kl}}(\nabla^h)_p(T_{\hat{g}})_{ij\overline{q}}&\leq 4\left[\lambda^{-1}\sqrt{g^{\overline{k}k}}\sqrt{g^{\overline{l}l}}\overline{\partial\alpha}_{\overline{kl}}\right]\left[\sqrt{g^{\overline{k}k}}\sqrt{g^{\overline{l}l}}(\nabla^h)_p(T_{\hat{g}})_{ij\overline{p}}\right]\\ &\leq 4\lambda^{-2}\left[\left|\partial\alpha\right|^2+\left|\nabla^hT_{\hat{g}}\right|_h\right]\\ &\leq C(\lambda,g_0,h)\Phi.\end{aligned}$$

Also we have

$$\begin{aligned}4g^{\overline{q}p}g^{\overline{k}i}g^{\overline{l}j}\overline{\partial\alpha}_{\overline{kl}}\Upsilon^r_{pi}(T_{\hat{g}})_{rj\overline{q}}&\leq 4\left(\sqrt{g^{\overline{p}p}}\sqrt{g^{\overline{k}k}}\Upsilon^r_{pk}\sqrt{h_{r\overline{r}}}\right)\left(\sqrt{g^{\overline{p}p}}\sqrt{g^{\overline{l}l}}\sqrt{h^{\overline{r}r}}(T_{\hat{g}})_{rl\overline{p}}\overline{\partial\alpha}_{\overline{kl}}\sqrt{g^{\overline{k}k}}\sqrt{g^{\overline{l}l}}\right)\\ &\leq\frac{1}{2}\left|\Upsilon\right|^2_{g^{-1},g^{-1},h}+C(\lambda,g_0,\hat{g})\left|\partial\alpha\right|^2\\ &\leq\frac{1}{2}\left|\Upsilon\right|^2_{g^{-1},g^{-1},h}+C(\lambda,g_0,\hat{g})\Phi.\end{aligned}$$

A similar estimate yields

$$\left\langle\partial\mu,\overline{\partial\alpha}\right\rangle\leq C(\lambda,g_0,\mu)\Phi.$$

Next we observe that, using the Cauchy-Schwarz inequality and the lower bound on g,

$$\begin{aligned} |T|^2 = & \ |\partial\omega|^2 \\ = & \ \left|\overline{\partial}\partial\alpha + \partial\hat{\omega}\right|^2 \\ \leq & \ 2\left|\overline{\nabla}\partial\alpha\right|^2 + 2\left|\partial\hat{\omega}\right|^2 \\ \leq & \ 2\left|\overline{\nabla}\partial\alpha\right| + C. \end{aligned}$$

Also we observe that since $Q \geq 0$ we have $-\langle h, Q\rangle \leq 0$, and $-\left\langle Q, \operatorname{tr}\partial\alpha \otimes \overline{\partial\alpha}\right\rangle \leq 0$. Collecting the above estimates yields

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi \leq C(\lambda, \hat{g}, h, \mu, g_0)\Phi.$$

By the maximum principle we conclude the result. □

6.2. **Proof of Theorem 1.1.** In this subsection we combine Theorem 1.7, Theorem 1.8, and further a priori estimates related to the Schwarz Lemma [46] and the Calabi-Yau Theorem [47] to yield the long time existence and convergence results of Theorem 1.1. To begin we record a rigidity result which will be used in the end of the proof of Theorem 1.1.

Lemma 6.3. *Let (M^{2n}, h, J) be a compact Hermitian manifold with $\rho(h) \leq 0$. Suppose g is a pluriclosed metric which is a steady soliton. Then g is a Calabi-Yau metric.*

Proof. As the metric is a soliton the pluriclosed evolution consists of pullback by a diffeomorphism generated by a vector field X. Thus by Lemma 6.1 we have

$$X \cdot \log\frac{\det g}{\det h} = \frac{\partial}{\partial t}\log\frac{\det g}{\det h} = \Delta\log\frac{\det g}{\det h} + |T|^2 - \operatorname{tr}_g\rho(h) \geq \Delta\log\frac{\det g}{\det h} + |T|^2.$$

It follows from the strong maximum principle that $|T|^2 = 0$, and therefore g is Kähler, and hence a compact steady Ricci soliton, which is an Einstein metric by [20]. Thus g is Kähler-Einstein. □

Lemma 6.4. *Let (M^{2n}, J, g_t) be a solution to pluriclosed flow. Suppose there exists a Hermitian metric h on M with nonpositive bisectional curvature. Then*

$$\sup_{M\times[0,T)} \operatorname{tr}_{g_t} h \leq \sup_M \operatorname{tr}_{g_0} h.$$

Proof. Inspecting the result of Lemma 6.2, we note that the matrix Q is positive semidefinite, hence $\langle h, Q\rangle \geq 0$. Also, by hypothesis h has nonpositive bisectional curvature, and hence $\Omega_h(g,g) \leq 0$. The result thus follows from the maximum principle. □

Lemma 6.5. *Let (M^{2n}, J) be a compact complex manifold admitting a Hermitian metric h of strictly negative bisectional curvature. Let δ denote the infimum of the absolute value of the bisectional curvatures of h. Let g_0 be a pluriclosed metric on M and let $\Lambda = [\sup_M \operatorname{tr}_{g_0} h]^{-1}$. The solution to pluriclosed flow with initial condition g_0 satisfies*

$$\sup_M \operatorname{tr}_{g_\tau} h \leq \frac{1}{\Lambda + \delta\tau}. \tag{6.1}$$

Proof. From Lemma 6.2 we yield,

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)\operatorname{tr}_g h = & \ -\left|\Upsilon(g,h)\right|^2_{g^{-1},g^{-1},h} - \langle h, Q\rangle + \Omega_h(g,g) \\ \leq & \ -\delta(\operatorname{tr}_g h)^2. \end{aligned}$$

Applying the maximum principle we conclude that $\sup_M \operatorname{tr}_{g_t} h$ is bounded above by the solution to the ODE

$$\frac{dF}{dt} = -\delta F^2, \qquad F(0) = \sup_M \operatorname{tr}_{g_0} h.$$

Solving this ODE yields the estimate (6.1). □

For the next two lemmas we require the 1-form reduction, and we specify the relevant background data in the case of a background metric of nonpositive bisectional curvature here. In particular, let h denote the given background metric of nonpositive bisectional curvature. Then in particular $\rho(h) \leq 0$. We furthermore set $\hat{\omega}_t = \omega_0 - t\rho(h) > 0$. In particular this means we set $\mu = 0$. With these choices made and an application of Lemma 3.2 we will assume a solution to (3.2) with respect to this background data.

Lemma 6.6. *Let (M^{2n}, J) be a compact complex manifold admitting a Hermitian metric h of strictly negative bisectional curvature. Let g_t denote a solution to the pluriclosed flow on (M, J), and let α_t denote the corresponding solution to (3.2). There exists a constant $C = C(g_0, h)$ such that for any smooth existence time $\tau > 0$, one has*

$$\sup_M |\partial \alpha_\tau|^2 \leq C.$$

Proof. Let

$$\Phi := 1 + \operatorname{tr}_{g_t} h + |\partial\alpha|^2 .$$

Combining Proposition 4.9 and Lemma 6.2 yields

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta\right)\Phi = &-|\Upsilon(g,h)|^2_{g^{-1},g^{-1},h} - \langle h, Q\rangle + \Omega_h(g,g)\\ &- |\nabla\partial\alpha|^2 - \left|\overline{\nabla}\partial\alpha\right|^2 - 2\left\langle Q, \operatorname{tr}\partial\alpha\otimes\overline{\partial\alpha}\right\rangle + 2\Re\sqrt{-1}\left\langle \operatorname{tr}_g \nabla^g \hat{T}, \overline{\partial\alpha}\right\rangle .\end{aligned}$$

All terms have a favorable sign except the last one. To estimate this we first of all express

$$\begin{aligned}\left\langle \operatorname{tr}_g \nabla^g \hat{T}, \overline{\partial\alpha}\right\rangle &= g^{\overline{q}p}\nabla_p \hat{T}_{ij\overline{q}}\overline{\partial\alpha}_{\overline{kl}} g^{\overline{k}i} g^{\overline{l}j}\\ &= g^{\overline{q}p} g^{\overline{k}i} g^{\overline{l}j}\overline{\partial\alpha}_{\overline{kl}}\left[\nabla^h_p \hat{T}_{ij\overline{q}} + \Upsilon(g,h)^r_{pi}\hat{T}_{rj\overline{q}} + \Upsilon(g,h)^r_{pj}\hat{T}_{ir\overline{q}}\right].\end{aligned}$$

Then we estimate in a basis where $h = \mathrm{Id}$ and g is diagonalized,

$$\begin{aligned}g^{\overline{q}p} g^{\overline{k}i} g^{\overline{l}j}\overline{\partial\alpha}_{\overline{kl}}\nabla^h_p\hat{T}_{ij\overline{q}} &\leq \frac{1}{(\Lambda+\delta t)^2}\left[\sqrt{g^{\overline{k}k}}\sqrt{g^{\overline{l}l}}\overline{\partial\alpha}_{\overline{kl}}\right]\left[\nabla^h_p \hat{T}_{kl\overline{p}}\right]\\ &\leq \frac{1}{(\Lambda+\delta t)^2}\left[|\partial\alpha|^2 + \left|\nabla^h\hat{T}\right|^2_h\right]\\ &\leq \frac{C\Phi}{(\Lambda+\delta t)^2}.\end{aligned}$$

Also we have

$$\begin{aligned}g^{\overline{q}p} g^{\overline{k}i} g^{\overline{l}j}\overline{\partial\alpha}_{\overline{kl}}\Upsilon^r_{pi}\hat{T}_{rj\overline{q}} &\leq \left(\sqrt{g^{\overline{p}p}}\sqrt{g^{\overline{k}k}}\Upsilon^r_{pk}\sqrt{h_{r\overline{r}}}\right)\left(\sqrt{g^{\overline{p}p}}\sqrt{g^{\overline{l}l}}\sqrt{h^{\overline{r}r}}\hat{T}_{rl\overline{p}}\overline{\partial\alpha}_{\overline{kl}}\sqrt{g^{\overline{k}k}}\sqrt{g^{\overline{l}l}}\right)\\ &\leq \frac{1}{2}|\Upsilon|^2_{g^{-1},g^{-1},h} + \frac{C}{(\Lambda+\delta t)^2}|\partial\alpha|^2\\ &\leq \frac{1}{2}|\Upsilon|^2_{g^{-1},g^{-1},h} + \frac{C\Phi}{(\Lambda+\delta t)^2}.\end{aligned}$$

Combining these inequalities yields the differential inequality

$$\left(\frac{\partial}{\partial t} - \Delta\right) \Phi \leq \frac{C\Phi}{(\Lambda + \delta t)^2}.$$

By the maximum principle, we obtain that

$$\sup_{M \times \{t\}} \Phi \leq F_t,$$

where

$$\frac{dF}{dt} = \frac{CF}{(\Lambda + \delta t)^2}, \qquad F_0 = \sup_{M \times \{0\}} \Phi.$$

This can be solved explicitly, with

$$F_t = F_0 \exp\left[\frac{C\delta^{-1}}{\Lambda} - \frac{C\delta^{-1}}{\Lambda + \delta t}\right].$$

The proposition follows. □

Lemma 6.7. *Let (M^{2n}, J) be a compact complex manifold admitting a Hermitian metric h of constant negative bisectional curvature $-\delta$. Given g_0 a pluriclosed metric there exists a constant $C = C(g_0, h)$ such that the solution to pluriclosed flow with initial condition g_0 satisfies*

$$C^{-1}(1 + \delta t) h \leq g_t \leq C(1 + \delta t) h.$$

Proof. Let

$$\Phi = \log \frac{\det g}{\det h} + 2\left(1 + \operatorname{tr}_{g_t} h + |\partial\alpha|^2\right).$$

Inspecting the proof of Lemma 6.6 yields the estimate

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(1 + \operatorname{tr}_{g_t} h + |\partial\alpha|^2\right) \leq \ -\frac{1}{2}\left|\Upsilon(g,h)\right|^2_{g^{-1},g^{-1},h} - \left|\overline{\nabla}\partial\alpha\right|^2 + \frac{C}{(\Lambda + \delta t)^2}.$$

Combining this estimate with Lemma 6.1 yields we yield

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi \leq \ |T|^2_g - \operatorname{tr}_g \rho(h) - \left|\Upsilon(g,h)\right|^2_{g^{-1},g^{-1},h} - 2\left|\overline{\nabla}\partial\alpha\right|^2 + \frac{C}{(\Lambda + \delta t)^2}.$$

First, since h has constant negative bisectional curvature δ, we have $\rho(h) = -n\delta h$ and estimate

$$-\operatorname{tr}_g \rho(h) = n\delta \operatorname{tr}_g h \leq \frac{n\delta}{\Lambda + \delta t}.$$

Moreover, we estimate

$$\begin{aligned}
|T|^2 = & \ \left|\partial\hat{\omega} + \overline{\nabla}\partial\alpha\right|^2 \\
& \leq 2\left|\partial\hat{\omega}\right|^2 + 2\left|\overline{\nabla}\partial\alpha\right|^2 \\
& \leq C\left(\operatorname{tr}_g h\right)^3 + 2\left|\overline{\nabla}\partial\alpha\right|^2 \\
& \leq \frac{C}{(\Lambda + \delta t)^3} + 2\left|\overline{\nabla}\partial\alpha\right|^2.
\end{aligned}$$

Combining these yields the estimate

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi \leq \frac{n\delta}{\Lambda + \delta t} + \frac{C}{(\Lambda + \delta t)^2}.$$

Applying the maximum principle yields

$$\sup_{M \times \{t\}} \Phi \leq \sup_{M \times \{0\}} \Phi + C + n \ln\left(\Lambda + \delta t\right).$$

Exponentiating yields

$$\frac{\det g}{\det h} \leq C\left(\Lambda + \delta t\right)^n.$$

By the arithmetic geometric mean and (6.1) we conclude

$$\operatorname{tr}_h g_t \leq (\operatorname{tr}_{g_t} h)^{n-1} \frac{\det g}{\det h} \leq C\left(\Lambda + \delta t\right).$$

The lemma follows. □

Proof of Theorem 1.1. First we establish statement (1). Using the metric h of nonpositive bisectional curvature we obtain a uniform lower bound for the metric from Lemma 6.4. The result now follows by applying Theorems 1.7 and 1.8.

Now we address statement (2). Let h denote a flat Kähler metric on (M^{2n}, J). Setting $\hat{g}_t = g_0$ and $\mu = 0$ we obtain a solution to $(\hat{g}, h, \mu)$-reduced pluriclosed flow via Lemma 3.2. We now claim that $[H_0] = 0$. By symmetrizing the Kähler form over the complex Lie group action on the torus we obtain a new left-invariant Kähler form. However a direct calculation shows that left-invariant Hermitian metrics on tori are automatically Kähler, and so have vanishing torsion class. Since symmetrization preserves cohomology classes this means that $[H_0] = 0$. By the $\partial\overline{\partial}$-lemma it follows that $[\partial\omega_0] = 0 \in H^{2,1}(M)$, thus we can choose η and ϕ according to Proposition 4.10. Using Lemma 6.4 we have an a priori lower bound for the metric along the flow. On the other hand using the result of Proposition 4.10 and Lemma 6.1 we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left[\log\frac{\det g}{\det h} + |\phi|^2\right] \leq -\left|\nabla\phi\right|^2 - 2\left\langle Q, \phi\otimes\overline{\phi}\right\rangle.$$

By applying the maximum principle we obtain an a priori upper bound for the metric along the flow. Theorem 1.7 then implies uniform C^∞ estimates for all times.

We now address the convergence at infinity. Using the uniform C^∞ bounds, any sequence of times $t_j \to \infty$ admits a smooth subsequential limiting metric on the same complex manifold. Moreover, these same uniform C^∞ estimates imply that the Perelman-type $\mathcal{F}$ functional for the pluriclosed flow ([38] Theorem 1.1) has a uniform upper bound for all times. It follows that any subsequential limit as described above is a pluriclosed steady soliton. From Lemma 6.3 it follows that this limiting metric g_∞ is a Kähler-Einstein metric. Each Kähler class on the torus admits a unique flat Kähler-Einstein metric, hence g_∞ is flat. It now follows from the linear/dynamic stability result of ([36] Theorem 1.2) that the whole flow converges exponentially to g_∞, as required.

Now we establish statement (3). By Lemma 6.7 we obtain that for any smooth interval of existence $[0, \tau)$ one has upper and lower bounds on the metric, and a uniform estimate of $|\partial\overline{\alpha}|^2$. It follows from Theorem 1.7 that the flow exists smoothly on $[0, \infty)$. We now translate these estimates to the normalized flow. In particular, if g_t denotes the solution to pluriclosed flow and we let

$$\breve{g}_s := e^{-s} g_{e^s},$$

then $\breve{g}_s$ is the unique solution to (2.2) with initial condition g_0. Moreover, setting $\breve{\alpha}_s = e^{-s}\alpha_{e^s}$, it follows that $\breve{\alpha}_t$ is a solution to the normalized 1-form flow. Observe from Lemma 6.7 that $\breve{g}$ has uniform upper and lower bounds. Also, from Lemma 6.6 there is a uniform upper bound for $|\partial\breve{\alpha}|^2$ due to its natural scaling invariance. From Theorem 1.7 we obtain uniform $C^{k,\alpha}$ estimates on g_t for all k, α and all $t > 0$. We now employ a blowdown argument to show convergence at infinity. Choose a sequence $t_j \to \infty$ and consider the sequence of solutions $\{g_j(t) = t_j^{-1} g(t_j t)\}$. By the estimates we have already shown there is a subsequence converging to a limit flow $g_\infty(t)$ defined on $(0, \infty)$. Using the monotone expanding entropy functional for pluriclosed flow ([38] Corollary 6.8), it follows that g_∞ is Kähler-Einstein, finishing the proof. □

7. Global existence and convergence results on generalized Kähler manifolds

7.1. **Setup.** In this section we exploit the estimates above to establish new long time existence results for the pluriclosed flow in the setting of commuting generalized Kähler geometry. We briefly recall here the discussion in [35] wherein the pluriclosed flow in the setting of generalized Kähler geometry with commuting complex structures is reduced to a fully nonlinear parabolic PDE.

7.1.1. *Differential geometric aspects.* Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold satisfying $[J_A, J_B] = 0$. Define

$$\Pi := J_A J_B \in \mathrm{End}(TM).$$

It follows that $\Pi^2 = \mathrm{Id}$, and Π is g-orthogonal, hence Π defines a g-orthogonal decomposition into its ± 1 eigenspaces, which we denote

$$TM = T_+ M \oplus T_- M.$$

Moreover, on the complex manifold (M^{2n}, J_A) we can similarly decompose the complexified tangent bundle $T^{1,0}_{\mathbb{C}}$. For notational simplicity we will denote

$$T^{1,0}_{\pm} := \ker\left(\Pi \mp I\right) : T^{1,0}_{\mathbb{C}}(M, J_A) \to T^{1,0}_{\mathbb{C}}(M, J_A).$$

We use similar notation to denote the pieces of the complex cotangent bundle. Other tensor bundles inherit similar decompositions. The one of most importance to us is

$$\begin{aligned}\Lambda^{1,1}_{\mathbb{C}}(M, J_A) = &\ \left(\Lambda^{1,0}_{+} \oplus \Lambda^{1,0}_{-}\right) \wedge \left(\Lambda^{0,1}_{+} \oplus \Lambda^{0,1}_{-}\right)\\ = &\ \left[\Lambda^{1,0}_{+} \wedge \Lambda^{0,1}_{+}\right] \oplus \left[\Lambda^{1,0}_{+} \wedge \Lambda^{0,1}_{-}\right] \oplus \left[\Lambda^{1,0}_{-} \wedge \Lambda^{0,1}_{+}\right] \oplus \left[\Lambda^{1,0}_{-} \wedge \Lambda^{0,1}_{-}\right].\end{aligned}$$

Given $\mu \in \Lambda^{1,1}_{\mathbb{C}}(M, J_A)$ we will denote this decomposition as

$$\mu := \mu^{+} + \mu^{\pm} + \mu^{\mp} + \mu^{-}. \tag{7.1}$$

These decompositions allow us to decompose differential operators as well. In particular we can express

$$d = d_+ + d_-, \qquad \partial = \partial_+ + \partial_-, \qquad \overline{\partial} = \overline{\partial}_+ + \overline{\partial}_-.$$

As explained in [35], the crucial differential operator governing the local generality of generalized Kähler metrics in this setting is

$$\square := \sqrt{-1}\left(\partial_+ \overline{\partial}_+ - \partial_- \overline{\partial}_-\right).$$

In particular, locally a generalized Kähler metric in this setting is in the image of $\square$.

7.1.2. *A characteristic class.*

Definition 7.1. Let (M^{2n}, J_A, J_B) be a bicomplex manifold such that $[J_A, J_B] = 0$. Let

$$\chi(J_A, J_B) = c_1^{+}(T^{1,0}_{+}) - c_1^{-}(T^{1,0}_{+}) + c_1^{-}(T^{1,0}_{-}) - c_1^{+}(T^{1,0}_{-}).$$

The meaning of this formula is the following: fix Hermitian metrics $h_{\pm}$ on the holomorphic line bundles $\det T^{1,0}_{\pm}$, and use these to define elements of $c_1(T^{1,0}_{\pm})$, and then project according to the decomposition (7.1). In particular, given such metrics $h_{\pm}$ we let $\rho(h_{\pm})$ denote the associated representatives of $c_1(T^{1,0}_{\pm})$, and then let

$$\chi(h_{\pm}) = \rho^{+}(h_{+}) - \rho^{-}(h_{+}) + \rho^{-}(h_{-}) - \rho^{+}(h_{-}).$$

This definition yields a well-defined class in a certain cohomology group, defined in [35], which we now describe.

Definition 7.2. Let (M^{2n}, J_A, J_B) be a bihermitian manifold with $[J_A, J_B] = 0$. Given $\phi_A \in \Lambda^{1,1}_{J_A,\mathbb{R}}$, let $\phi_B = -\phi_A(\Pi\cdot, \cdot) \in \Lambda^{1,1}_{J_B,\mathbb{R}}$. We say that ϕ_A is *formally generalized Kähler* if

$$
\begin{aligned}
d^c_{J_A}\phi_A &= -\, d^c_{J_B}\phi_B,\\
dd^c_{J_A}\phi_A &= 0.
\end{aligned}
\tag{7.2}
$$

Definition 7.3. Let (M^{2n}, g, J_A, J_B) denote a generalized Kähler manifold such that $[J_A, J_B] = 0$. Let

$$
\mathcal{H} := \frac{\left\{\phi_A \in \Lambda^{1,1}_{J_A,\mathbb{R}} \mid \phi_A \text{ satisfies (7.2)}\right\}}{\{\Box f \mid f \in C^\infty(M)\}}.
$$

As shown in [35], the operator χ yields a well-defined class in $\mathcal{H}$, analogous to the first Chern class of a Kähler manifold. With this kind of cohomology space, we can define the analogous notion to the "Kähler cone," which we refer to as $\mathcal{P}$, the "positive cone."

Definition 7.4. Let (M^{2n}, g, J_A, J_B) denote a generalized Kähler manifold such that $[J_A, J_B] = 0$. Let

$$
\mathcal{P} := \left\{[\phi] \in \mathcal{H} \mid \exists\, \omega \in [\phi], \omega > 0\right\}.
$$

Definition 7.5. Let (M^{2n}, g, J_A, J_B) be a generalized Kähler manifold such that $[J_A, J_B] = 0$. We say that $\chi = \chi(J_A, J_B) > 0$, (resp. $(\chi < 0,\ \chi = 0)$ if $\chi \in \mathcal{P}$, (resp. $-\chi \in \mathcal{P}, \chi = 0$).

7.1.3. *Pluriclosed flow in commuting generalized Kähler geometry.* With this setup we describe how to reduce (1.1) to a scalar PDE in the setting of commuting generalized Kähler manifolds. First we recall that it follows from ([35] Proposition 3.2, Lemma 3.4) that the pluriclosed flow in this setting reduces to

$$
\frac{\partial}{\partial t}\omega = -\,\chi(\omega). \tag{7.3}
$$

From the discussion above, we see that a solution to (7.3) induces a solution to an ODE in $\mathcal{P}$, namely

$$
[\omega_t] = [\omega_0] - t\chi.
$$

Now we can make a definition which specializes Definition 2.7 to this setting.

Definition 7.6. Given (M^{2n}, g, J_A, J_B) a generalized Kähler manifold with $[J_A, J_B] = 0$, let

$$
\tau^*(g) := \sup\left\{t \geq 0 \mid [\omega] - t\chi \in \mathcal{P}\right\}.
$$

Now fix $\tau < \tau^*$, so that by hypothesis if we fix arbitrary metrics $\widetilde{h}_\pm$ on $T^{1,0}_\pm$, there exists $a \in C^\infty(M)$ such that

$$
\omega_0 - \tau\chi(\widetilde{h}_\pm) + \Box a > 0.
$$

Now set $h_\pm = e^{\pm\frac{a}{2\tau}}\widetilde{h}_\pm$. Thus $\omega_0 - \tau\chi(h_\pm) > 0$, and by convexity it follows that

$$
\hat{\omega}_t := \omega_0 - t\chi(h_\pm) > 0
$$

is a smooth one-parameter family of generalized Kähler metrics. Furthermore, given a function $f \in C^\infty(M)$, let

$$
\omega_f := \hat{\omega} + \Box f,
$$

with g^f the associated Hermitian metric. Now suppose that u_t satisfies

$$
\frac{\partial}{\partial t}u = \log\frac{\det g^u_+ \det h_-}{\det h_+ \det g^u_-}. \tag{7.4}
$$

An elementary calculation using the transgression formula for the first Chern class ([35] Lemma 3.4) yields that ω_u solves (7.3). This reduction can be refined in the case $\chi = 0$, as we will detail below.

7.2. **Long time existence.**

Proof of Theorem 1.2. First, we choose two background metrics using the given topological hypotheses. In particular, since $\chi(J_A, J_B) = 0$ we may choose a Hermitian metric h such that $\chi(h_\pm) = 0$. Also, since $c_1^{BC}(J_A) \leq 0$, we may choose a Hermitian metric $\hat{h}$ such that $\rho(\hat{h}) \leq 0$. Following the discussion in §7.1, we can reduce the pluriclosed flow in this setting to a scalar PDE. In this case the background metric is $\hat{g}_t = g_0$, and then setting $g^u = g_0 + \Box u_t$, we let u solve

$$\frac{\partial}{\partial t} u = \log \frac{\det g^u_+ \det h_-}{\det h_+ \det g^u_-}, \qquad u_0 \equiv 0. \tag{7.5}$$

Using that $\chi(h_\pm) = 0$ it follows as described above that g^u_t is the solution to pluriclosed flow with initial condition g_0.

We now derive a priori estimates for g^u_t. Using ([35] Lemma 4.2), we observe the evolution equation

$$\left(\frac{\partial}{\partial t} - \Delta \right) \dot{u} = 0.$$

It follows from the maximum principle that for any smooth existence time τ we have

$$\sup_{M \times \{\tau\}} \left| \dot{u} \right| \leq \sup_{M \times \{0\}} \left| \dot{u} \right|. \tag{7.6}$$

A direct integration in time then shows that there is a constant depending the initial data such that

$$\sup_{M \times \{\tau\}} \left| u \right| \leq C \left(1 + \tau \right). \tag{7.7}$$

Also, from Lemma 6.1 and the fact that $\rho(\hat{h}) \leq 0$ we observe that

$$\left(\frac{\partial}{\partial t} - \Delta \right) \log \frac{\det g^u}{\det \hat{h}} = \left| T \right|^2 - \operatorname{tr}_g \rho(\hat{h}) \geq 0.$$

Thus from the maximum principle we conclude that

$$\inf_{M \times \{\tau\}} \log \frac{\det g^u}{\det \hat{h}} \geq \inf_{M \times \{0\}} \log \frac{\det g^u}{\det \hat{h}}. \tag{7.8}$$

Combining (7.6) and (7.8) and some elementary identities yields that there is a constant C depending on the background data such that

$$\log \frac{\det g^u_-}{\det h_-} \geq -C. \tag{7.9}$$

Since $\operatorname{rank} T^{1,0}_- = 1$ we conclude that

$$g^u_- \geq e^{-C} h_-. \tag{7.10}$$

Next we aim to derive a lower bound for g_+. To do that we fix some $\lambda \in \mathbb{R}$ and set

$$\Phi = \log \operatorname{tr}_{g_+} h_+ - \lambda u.$$

Using ([35] Lemma 4.3) we obtain

$$\left(\frac{\partial}{\partial t}-\Delta\right)\Phi = \frac{1}{\operatorname{tr}_{g_+} h_+}\left[-\left|\Upsilon(g_+,h_+)\right|^2_{g^{-1},g^{-1},h}+\frac{\left|\nabla \operatorname{tr}_{g_+} h_+\right|^2_g}{\operatorname{tr}_{g_+} h_+}-\operatorname{tr} g_+^{-1}h_+g_+^{-1}Q+g^{\bar{q}p}g^{\overline{\beta}_+\alpha_+}\Omega^{h_+}_{p\bar{q}\alpha_+\overline{\beta}_+}\right]$$
$$-\lambda\dot{u}+\lambda\Delta u.$$

We estimate the various terms of this equation. First,using (7.10) we have

$$\begin{aligned}
\frac{1}{\operatorname{tr}_{g_+} h_+}g^{\bar{q}p}g^{\overline{\beta}_+\alpha_+}\Omega^{h_+}_{p\bar{q}\alpha_+\overline{\beta}_+} \leq&\ \frac{C}{\operatorname{tr}_{g_+} h_+}g^{\bar{q}p}g^{\overline{\beta}_+\alpha_+}h_{p\bar{q}}h_{\alpha_+\overline{\beta}_+}\\
=&\ C\operatorname{tr}_g h\\
=&\ C\left[\operatorname{tr}_{g_+} h_+ + \operatorname{tr}_{g_-} h_-\right]\\
\leq&\ C\operatorname{tr}_{g_+} h_+ + C.
\end{aligned}$$

Next we estimate using the Cauchy-Schwarz inequality, computing in coordinates where at the point in question $h_+=\mathrm{Id}$ and g is diagonal,

$$\begin{aligned}
\frac{\left|\nabla \operatorname{tr}_{g_+} h_+\right|^2}{\operatorname{tr}_{g_+} h_+} =&\ \left(\sum g_+^{\bar{i}i}\right)^{-1} g^{\bar{j}j}\nabla_j \operatorname{tr}_{g_+} h_+ \nabla_{\bar{j}}\operatorname{tr}_{g_+} h_+\\
=&\ \left(\sum g_+^{\bar{i}i}\right)^{-1} g^{\bar{j}j}\left[-g_+^{\bar{l}m}g^+_{m\bar{p},j}g_+^{\bar{p}q}h_{q\bar{l}}+g^{\bar{l}m}h_{m\bar{l},j}\right]\left[-g_+^{\bar{r}s}g^+_{s\bar{t},\bar{j}}g_+^{\bar{t}u}h_{u\bar{r}}+g_+^{\bar{r}s}h_{s\bar{r},\bar{j}}\right]\\
=&\ \left(\sum g_+^{\bar{i}i}\right)^{-1} g^{\bar{j}j}\left[g_+^{\bar{l}m}\Upsilon(g_+,h_+)_{jm}^q h_{q\bar{l}}\right]\left[g_+^{\bar{r}s}\overline{\Upsilon}(g_+,h_+)_{\bar{j}\bar{r}}^{\bar{t}}h_{s\bar{t}}\right]\\
=&\ \left(\sum g_+^{\bar{i}i}\right)^{-1}\left[\left(g_+^{\bar{l}l}\right)^{\frac{1}{2}}\left[\left(g^{\bar{j}j}g_+^{\bar{l}l}\right)^{\frac{1}{2}}\Upsilon(g_+,h_+)_{jm}^l\right]\right]\left[\left(g_+^{\bar{r}r}\right)^{\frac{1}{2}}\left(g^{\bar{j}j}g_+^{\bar{r}r}\right)^{\frac{1}{2}}\overline{\Upsilon}(g_+,h_+)_{\bar{j}\bar{r}}^{\bar{s}}\right]\\
\leq&\ \left(\sum g_+^{\bar{i}i}\right)^{-1}\left[\sum g_+^{\bar{l}l}\right]^{\frac{1}{2}}\left[\left|\Upsilon(g_+,h_+)\right|^2_{g^{-1},g^{-1},h}\right]^{\frac{1}{2}}\left[\sum g_+^{\bar{r}r}\right]^{\frac{1}{2}}\left[\left|\Upsilon(g_+,h_+)\right|^2_{g^{-1},g^{-1},h}\right]^{\frac{1}{2}}\\
=&\ \left|\Upsilon(g_+,h_+)\right|^2_{g^{-1},g^{-1},h}.
\end{aligned}$$

Also, since $Q\geq 0$ we have $\operatorname{tr} g_+^{-1}h_+g_+^{-1}Q\geq 0$. Lastly, using the definition of g^u we observe that

$$\begin{aligned}
\Delta u =&\ g_u^{\overline{\beta}_+\alpha_+}u_{,\alpha_+\overline{\beta}_+}+g_u^{\overline{\beta}_-\alpha_-}u_{,\alpha_-\overline{\beta}_-}\\
=&\ g_u^{\overline{\beta}_+\alpha_+}\left(g^u_{\alpha_+\overline{\beta}_+}-g^0_{\alpha_+\overline{\beta}_+}\right)+g_u^{\overline{\beta}_-\alpha_-}\left(g^0_{\alpha_-\overline{\beta}_-}-g^u_{\alpha_-\overline{\beta}_-}\right)\\
=&\ \operatorname{rank} T^{1,0}_+ - \operatorname{rank} T^{1,0}_- - \operatorname{tr}_{g_+^u} g_+^0 + \operatorname{tr}_{g_-^u} g_-^0\\
\leq&\ -\operatorname{tr}_{g_+^u} g_+^0 + C\\
\leq&\ -c\operatorname{tr}_{g_+} h_+ + C,
\end{aligned}$$

for some constants c,C. Combining the above estimates together with (7.6), (7.7), and choosing λ sufficiently large yields

$$\begin{aligned}
\left(\frac{\partial}{\partial t}-\Delta\right)\Phi \leq&\ C\lambda+(C-\lambda c)\operatorname{tr}_{g_+^u} h_+.\\
\leq&\ C\lambda-\frac{\lambda}{2}\operatorname{tr}_{g_+} h_+\\
=&\ C\lambda-\frac{\lambda}{2}e^{\lambda u+\Phi}\\
\leq&\ C\lambda-\frac{\lambda}{2}e^{-\lambda C(1+t)+\Phi}.
\end{aligned}$$

At a spacetime maximum where $\Phi \geq \lambda C(1+t) + \log 2C$, we yield $(\frac{\partial}{\partial t} - \Delta)\Phi \leq 0$. It follows from the maximum principle that there is a constant C depending on the initial data such that

$$\sup_{M \times \{t\}} \Phi \leq C\,(1+t)\,.$$

From the definition of Φ and the a priori estimate for u this implies a lower bound for g_+ on any finite time interval. The theorem now follows from ([35] Proposition 5.3, [40] Theorem 1.2). □

7.3. **Yau's oscillation estimate on generalized Kähler manifolds.** In this subsection we adapt Yau's potential oscillation estimate [47] to the setting of commuting generalized Kähler geometry. We use ideas of Cherrier [8] in a similar way as exploited by Tosatti-Weinkove [43, 44], who proved a direct generalization of Yau's estimate in the Hermitian setting.

Theorem 7.7. *Let (M^{2n}, g, J_A, J_B) be a compact generalized Kähler manifold, and let $u \in C^\infty(M)$ satisfy*

$$\begin{aligned}
\left(\omega_+ + \sqrt{-1}\partial_+\overline{\partial}_+ u\right)^k &= e^{F_+}\omega_+^k,\\
\left(\omega_- - \sqrt{-1}\partial_-\overline{\partial}_- u\right)^l &= e^{F_-}\omega_-^l,\\
\operatorname{tr}_\omega \sqrt{-1}\partial\overline{\partial} u &> -\lambda.
\end{aligned}$$

There exists a constant $C = C(\sup F_+, \inf F_-, \lambda)$ such that

$$\operatorname{osc}_M u = \sup_M u - \inf_M u \leq C.$$

Remark 7.8. Usually in this type of estimate it is only an upper bound on the volume form which is required. Due to the extra minus sign appearing in the definition of ω_-^u the estimate depends on a lower bound for this partial volume form. Also note that the third hypothesis of Theorem 7.7, the lower bound on the Laplacian of u, is satisfied automatically in the Kähler setting assuming u defines a Kähler form, i.e. $\omega + \sqrt{-1}\partial\overline{\partial} u > 0$. It is an interesting challenge to remove this hypothesis.

To begin we record a certain estimate inspired by a lemma of Cherrier [8].

Lemma 7.9. *There exist C, p_0 so that for all $p \geq p_0$ we have*

$$\int_M \left|\partial e^{-\frac{p}{2}u}\right|_g^2 \omega_+^k \wedge \omega_-^l \leq Cp \int_M e^{-pu}\omega_+^k \wedge \omega_-^l.$$

Proof. We set

$$\mu_+ = \sum_{i=0}^{k-1} \omega_{u,+}^i \wedge \omega_+^{k-i-1}, \qquad \mu_- = \sum_{i=0}^{l-1} \omega_{u,-}^i \wedge \omega_-^{l-i-1}.$$

Observe by the generalized Kähler conditions that $d_\pm \mu_\pm = 0$. Using the bounds on the volume forms and integrating by parts yields

(7.11)
$$\begin{aligned}
C\int_M e^{-pu}\omega_+^k \wedge \omega_-^l \geq&\ \int_M e^{-pu}\left(\omega_{u,+}^k - \omega_+^k\right) \wedge \omega_-^l\\
=&\ \int_M e^{-pu}\sqrt{-1}\partial_+\overline{\partial}_+ u \wedge \mu_+ \wedge \omega_-^l\\
=&\ p\int_M e^{-pu}\sqrt{-1}\partial_+ u \wedge \overline{\partial}_+ u \wedge \mu_+ \wedge \omega_-^l + \int_M e^{-pu}\sqrt{-1}\overline{\partial}_+ u \wedge \mu_+ \wedge \partial_+\omega_-^l.
\end{aligned}$$

Next, using the Cauchy-Schwarz inequality one obtains

$$
\begin{aligned}
&\left| \frac{\sqrt{-1}\overline{\partial}_+ u \wedge \omega_{u,+}^i \wedge \omega_+^{k-i-1} \wedge \partial_+\omega_- \wedge \omega_-^{l-1}}{\omega_+^k \wedge \omega_-^l} \right| \\
&\qquad \leq \frac{C}{\epsilon} \frac{\sqrt{-1}\partial_+ u \wedge \overline{\partial}_+ u \wedge \omega_{u,+}^i \wedge \omega_+^{k-i-1} \wedge \omega_-^l}{\omega_+^k \wedge \omega_-^l} + C\epsilon \frac{\omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l}{\omega_+^k \wedge \omega_-^l}.
\end{aligned} \tag{7.12}
$$

Using this we have

$$
\begin{aligned}
-\int_M e^{-pu}\sqrt{-1}\overline{\partial}_+ u \wedge \mu_+ \wedge \partial_+\omega_-^l =& -l \int_M e^{-pu}\sqrt{-1}\overline{\partial}_+ u \wedge \left(\sum_{i=0}^{k-1} \omega_{u,+}^i \wedge \omega_+^{k-i-1} \right) \wedge \partial_+\omega_- \wedge \omega_-^{l-1} \\
&\leq \frac{C}{\epsilon} \sum_{i=0}^{k-1} \int_M e^{-pu}\sqrt{-1}\partial_+ u \wedge \overline{\partial}_+ u \wedge \omega_{u,+}^i \wedge \omega_+^{k-i-1} \wedge \omega_-^l \\
&\quad + \epsilon C \sum_{i=0}^{k-1} \int_M e^{-pu} \omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l.
\end{aligned}
$$

Now choose ϵ sufficiently small with respect to controlled constants and choose p_0 sufficiently large to obtain

$$
\begin{aligned}
-\int_M e^{-pu}\sqrt{-1}\overline{\partial}_+ u \wedge \mu_+ \wedge \partial_+\omega_-^l \leq& \frac{p}{4} \int_M e^{-pu}\sqrt{-1}\partial_+ u \wedge \overline{\partial}_+ u \wedge \mu_+ \wedge \omega_-^l \\
&+ \epsilon C \int_M e^{-pu} \omega_+^k \wedge \omega_-^l + \epsilon C \sum_{i=1}^{k-1} \int_M e^{-pu} \omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l.
\end{aligned} \tag{7.13}
$$

Combining (7.11), (7.13) we get

$$
\frac{p}{2} \int_M e^{-pu}\sqrt{-1}\partial_+ u \wedge \overline{\partial}_+ u \wedge \mu_+ \leq C \int_M e^{-pu} \omega_+^k \wedge \omega_-^l + \epsilon C \sum_{i=1}^{k-1} \int_M e^{-pu} \omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l. \tag{7.14}
$$

Next we claim that for $j = 0, \dots, k$ there exist constants C_j such that

$$
\begin{aligned}
&\frac{p}{2^{j+1}} \int_M e^{-pu}\sqrt{-1}\partial_+ u \wedge \overline{\partial}_+ u \wedge \mu_+ \wedge \omega_-^l \\
&\qquad \leq C_j \int_M e^{-pu} \omega_+^k \wedge \omega_-^l + \epsilon C_j \sum_{i=1}^{k-j} \int_M e^{-pu} \omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l.
\end{aligned} \tag{7.15}
$$

We prove this by induction on j, the case $j = 0$ being equivalent to (7.14). We assume the result for a general j and show it for $j + 1$. First we separate the last term of (7.15) to yield

$$
\begin{aligned}
\epsilon C_j \sum_{i=1}^{k-j} \int_M e^{-pu} \omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l =& \epsilon C_j \sum_{i=1}^{k-j} \int_M e^{-pu} \omega_{u,+}^{i-1} \wedge \omega_+^{k-i+1} \wedge \omega_-^l \\
&+ \epsilon C_j \sum_{i=1}^{k-j} \int_M e^{-pu} \sqrt{-1}\partial_+\overline{\partial}_+ u \wedge \omega_{u,+}^{i-1} \wedge \omega_+^{k-i} \wedge \omega_-^l \\
=:& \, I + II.
\end{aligned} \tag{7.16}
$$

The term I can be included on the right hand side of (7.15) in showing the inductive step. For term II we further integrate by parts

$$\begin{aligned} II = {} & \epsilon C_j \sum_{i=0}^{k-j-1} \Big[p \int_M e^{-pu} \sqrt{-1} \partial_+ u \wedge \overline{\partial}_+ u \wedge \omega_{u,+}^i \wedge \omega_+^{k-i-1} \wedge \omega_-^l \\ & \qquad - \int_M e^{-pu} \sqrt{-1} \overline{\partial}_+ u \wedge \omega_{u,+}^i \wedge \omega_+^{k-i-1} \wedge \partial_+ \omega_-^l \Big] \\ = {} & II_A + II_B. \end{aligned}$$

Choosing ϵ sufficiently small yields

$$II_A \leq \frac{p}{2^{j+3}} \int_M e^{-pu} \sqrt{-1} \partial_+ u \wedge \overline{\partial}_+ u \wedge \mu_+ \wedge \omega_-^l. \tag{7.17}$$

Next using (7.12) we have

$$\begin{aligned} II_B \leq {} & \epsilon C C_j \sum_{i=0}^{k-j-1} \Big[\int_M e^{-pu} \sqrt{-1} \partial_+ u \wedge \overline{\partial}_+ u \wedge \omega_{u,+}^i \wedge \omega_+^{k-i-1} \wedge \omega_-^l \\ & \qquad + \int_M e^{-pu} \omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l \Big] \\ \leq {} & \frac{p}{2^{j+3}} \int_M e^{-pu} \sqrt{-1} \partial_+ u \wedge \overline{\partial}_+ u \wedge \mu_+ \wedge \omega_-^l + \epsilon C_j \sum_{i=0}^{k-j-1} \int_M e^{-pu} \omega_{u,+}^i \wedge \omega_+^{k-i} \wedge \omega_-^l, \end{aligned} \tag{7.18}$$

where the last line follows by ensuring ϵ is chosen sufficiently small. Plugging (7.17) and (7.18) into (7.16) finishes the proof of (7.15), which for $j = k$ can be rewritten as

$$\int_M \left| \partial_+ e^{-\frac{p}{2} u} \right|_g^2 \omega_+^k \wedge \omega_-^l \leq C p \int_M e^{-pu} \omega_+^k \wedge \omega_-^l. \tag{7.19}$$

Arguing similarly and using a *lower* bound for the partial volume form $\omega_{u,-}^l$ we have

$$\begin{aligned} C \int_M e^{-pu} \omega_+^k \wedge \omega_-^l \geq {} & \int_M e^{-pu} \omega_+^k \wedge \left(\omega_-^l - \omega_{u,-}^l \right) \\ = {} & \int_M e^{-pu} \omega_+^k \wedge \sqrt{-1} \partial_- \overline{\partial}_- u \wedge \mu_- \\ = {} & p \int_M e^{-pu} \sqrt{-1} \partial_- u \wedge \overline{\partial}_- u \wedge \mu_- \wedge \omega_+^k + \int_M e^{-pu} \sqrt{-1} \overline{\partial}_- u \wedge \mu_- \wedge \partial_- \omega_+^k. \end{aligned}$$

A similar series of estimates as detailed above yields the inequality

$$\int_M \left| \partial_- e^{-\frac{p}{2} u} \right|_g^2 \omega_+^k \wedge \omega_-^l \leq C p \int_M e^{-pu} \omega_+^k \wedge \omega_-^l. \tag{7.20}$$

Combining (7.19) and (7.20) yields the result. □

Lemma 7.10. *There exists a constant C such that, if we set $v = u - \inf_M u$,*

$$||v||_{L^1} \leq ||v - \underline{v}||_{L^1} + C.$$

Proof. Combining Lemma 7.9 with the Sobolev inequality, we see that for $p \geq p_0$ and $\lambda = \frac{n}{n-1}$ we have

$$\begin{aligned} \left|\left|e^{-u}\right|\right|_{L^{\lambda p}} &= \left(\int_M e^{-\lambda p u}\omega^n\right)^{\frac{1}{\lambda p}} \\ &\leq \left[C\int_M \left|\partial e^{-\frac{p}{2}u}\right|^2 \omega^n + C\int_M e^{-pu}\omega^n\right]^{\frac{1}{p}} \\ &\leq C^{\frac{1}{p}} p^{\frac{1}{p}} \left|\left|e^{-u}\right|\right|_{L^p}. \end{aligned}$$

Iterating this estimate yields

$$\begin{aligned} \left|\left|e^{-u}\right|\right|_{L^\infty} &= \lim_{l\to\infty} \left|\left|e^{-u}\right|\right|_{L^{p_0\lambda^l}} \\ &\leq \lim_{l\to\infty} C^{\sum_{i=0}^{l}\lambda^{-i}} \prod_{i=0}^{l-1} \left(\lambda^i p_0\right)^{\frac{1}{\lambda^i p_o}} \left|\left|e^{-u}\right|\right|_{L^{p_0}} \\ &\leq C_0 \left|\left|e^{-u}\right|\right|_{L^{p_0}}. \end{aligned} \tag{7.21}$$

We use this in conjunction with an argument from ([44] Lemma 3.2) to show that there exist uniform constants C_1 and δ so that

$$\left|\{u < \inf_M u + C_1\}\right| \geq \delta. \tag{7.22}$$

Let $w = p_0 u$, so that (7.21) reads

$$e^{-\inf w} \leq C_0 \int_M e^{-w}\omega^n. \tag{7.23}$$

Now we split

$$\int_M e^{-w} = \int_{\left\{e^{-w}\geq\frac{1}{2}\int_M e^{-w}\right\}} e^{-w} + \int_{\left\{e^{-w}<\frac{1}{2}\int_M e^{-w}\right\}} e^{-w} = I + II.$$

Since

$$\left|\{e^{-w} \geq \frac{1}{2}\int_M e^{-w}\}\right| \leq \left|\{e^{-w} \geq \frac{1}{2C_0}e^{-\inf w}\}\right| = \left|\{w < \inf_M w + C_1\}\right|,$$

we obtain, combining with (7.23),

$$I \leq \left|\{w < \inf_M w + C_1\}\right| \sup_M e^{-w} \leq C_0 \left|\{w < \inf_M w + C_1\}\right| \int_M e^{-w}.$$

On the other hand we directly estimate using that $\int_M d\mu = 1$,

$$II \leq \frac{1}{2}\int_M e^{-w}.$$

Thus we obtain

$$\int_M e^{-w} \leq \left[C_0 \left|\{w < \inf_M w + C_1\}\right| + \frac{1}{2}\right] \int_M e^{-w}$$

Rearranging this yields

$$\left|\{w < \inf_M w + C_1\}\right| \geq \delta$$

for some uniform constant $\delta > 0$. Since p_0 is some fixed constant, we obtain (7.22). Using this we have

$$\begin{aligned}||v||_{L^1} &= \int_M v d\mu \\ &= \underline{v} \\ &\leq \delta^{-1} \int_{\{v \leq C_1\}} \underline{v} \\ &\leq \int_{\{v \leq C_1\}} |v - \underline{v}| + C_1 \\ &\leq ||v - \underline{v}||_{L^1} + C_1.\end{aligned}$$

□

Lemma 7.11. *(*[43] *Lemma 2.3) Let (M^{2n}, ω_G, J) be a compact complex manifold with Gauduchon metric ω_G. Let $f \in C^\infty(M)$ satisfy*

$$\Delta_{\omega_G} f \geq -C_0.$$

Then there exist constants C_1, C_2 depending on (M^{2n}, ω_G, J) and C_0 such that for all $p \geq 1$,

$$\int_M \left| \partial f^{\frac{p+1}{2}} \right|^2_{\omega_G} \omega_G^n \leq C_1 p \int_M f^p \omega_g^n, \tag{7.24}$$

and

$$\sup_M f \leq C_2 \max \left\{ \int_M f \omega_G^n, 1 \right\}. \tag{7.25}$$

Proof of Theorem 7.7. Let $v = u - \inf_M u$. By Gauduchon's theorem [16] ω admits a conformally related Gauduchon metric $\omega_G = e^\phi \omega$, for some smooth function ϕ. By assumption we have $\Delta_{\omega_G} v = e^{-\phi} \Delta_\omega v > -C$. Thus, by Lemma 7.11, it suffices to estimate the L^1 norm of v. Using (7.24) with $p = 1$, the Poincaré inequality for ω_G, and Lemma 7.10, we have

$$\begin{aligned}||v||_{L^1} &\leq C + ||v - \underline{v}||_{L^1} \\ &\leq C + ||v - \underline{v}||_{L^2} \\ &\leq C + C\,||\partial v||_{L^2} \\ &\leq C + C\,||v||_{L^1}^{\frac{1}{2}}.\end{aligned}$$

The estimate for $||v||_{L^1}$, and hence the theorem, follows. □

7.4. **Convergence.** We now establish the convergence claims of Theorem 1.3. Before getting to the proof we record a topological splitting result for Kähler-Einstein manifolds.

Theorem 7.12. *(*[2, 3, 23]*) Let (M, J) be a compact complex manifold which admits a Kähler-Einstein metric g, and whose tangent bundle splits as a direct sum of two holomorphic sub-bundles $T_\pm M$. Then $T_\pm M$ are parallel with respect to the Levi-Civita connection of g. In particular, (M, g, J) is a local Kähler product of two Kähler-Einstein manifolds tangent to $T_\pm M$.*

Proof. Using the Bochner-Kodaira identity shows that for a holomorphic endomorphism Q of TM one has for a Kähler-Einstein background,

$$\int_M |\nabla Q| = \int_M \left\langle [g^{-1} \operatorname{Rc}, Q], Q \right\rangle = \int_M \langle [\lambda I, Q], Q \rangle = 0.$$

Thus Q is parallel, and the theorem follows from the de Rham decomposition theorem. □

Proof of Theorem 1.3. To begin we establish more refined versions of the estimates of Theorem 1.2. In particular we claim uniform equivalence of the metrics and uniform C^∞ estimates for all time. As in Theorem 1.2 we choose a Hermitian metric h such that $\chi(h_\pm) = 0$. Furthermore since now $c_1^{BC}(J_A) = 0$ we choose a Hermitian metric $\hat{h}$ such that $\rho(\hat{h}) = 0$. We now consider two reductions of the pluriclosed flow. First, following Theorem 1.2 we set $\hat{g} = g_0$, let $g^u = g_0 + \square u$ and let u solve

$$\frac{\partial}{\partial t} u = \log \frac{\det g_+^u \det h_-}{\det h_+ \det g_-^u}, \qquad u_0 \equiv 0.$$

On the other hand, with the choices $\hat{g}, \hat{h}$, and $\mu = 0$, using Lemma 3.2 we choose a solution α_t to $(\hat{g}_t, \hat{h}, 0)$-reduced pluriclosed flow. Since by hypothesis $[\partial \hat{\omega}_0] = 0$, there exists $\eta \in \Lambda^{2,0}$ such that

$$\partial \hat{\omega}_t = \partial \hat{\omega}_0 = \overline{\partial} \eta.$$

Thus choose $\phi = \partial \alpha - \eta$ and by Proposition 4.10 we have

$$\left(\frac{\partial}{\partial t} - \Delta_{g_t} \right) |\phi|^2 = - |\nabla \phi|^2 - |T|^2 - 2 \left\langle Q, \phi \otimes \overline{\phi} \right\rangle. \tag{7.26}$$

This differential inequality is very helpful in obtaining estimates of the metric, and is the reason for considering the α-reduction of the pluriclosed flow as well as the scalar reduction. While of course these two reductions are related, the scalar reduction involves also the background metric h, and the nature of these background terms interferes with simply using $\partial_- \partial_+ u$ as the torsion potential.

First we observe that the estimates (7.6), (7.8) and (7.9) of Theorem 1.2 still hold in this setting. Combining Lemma 6.1 with (7.26) yields

$$\left(\frac{\partial}{\partial t} - \Delta \right) \left[\log \frac{\det g^u}{\det \hat{h}} + |\phi|^2 \right] \leq 0.$$

It follows from the maximum principle that

$$\sup_{M \times \{t\}} \left[\log \frac{\det g^u}{\det \hat{h}} + |\phi|^2 \right] \leq C. \tag{7.27}$$

Combining (7.27) with (7.6) and (7.8) and making elementary manipulations yields

$$\left| \log \frac{\det g_\pm^u}{\det h_\pm} \right| \leq C. \tag{7.28}$$

Since $\operatorname{rank} T_-^{1,0} = 1$ we conclude that

$$C^{-1} h_- \leq g_-^u \leq C h_-. \tag{7.29}$$

Note that by combining the basic fact $g_+^u > 0$ with $g_-^u \leq C h_-$ we conclude that there is a constant C such that

$$\operatorname{tr}_{g_0} \sqrt{-1} \partial \overline{\partial} u > -C.$$

We now construct a normalized potential for the metric. In particular, we let v_t solve

$$\frac{\partial}{\partial t} v = \log \frac{\det g_+^u \det h_-}{\det h_+ \det g_-^u} - \frac{\int_M \log \frac{\det g_+^u \det h_-}{\det h_+ \det g_-^u} \omega_0^n}{\int_M \omega_0^n}, \qquad v_0 \equiv 0.$$

Certainly v_t only differs from u_t by a constant, thus $g^u = g^v$. The estimates (7.28) above imply

$$\left| \log \frac{\det g_\pm^v}{\det h_\pm} \right| \leq C, \qquad \operatorname{tr}_{g_0} \sqrt{-1} \partial \overline{\partial} v > -C. \tag{7.30}$$

Moreover, an elementary calculation shows that for all t,

$$\int_M v \omega_0^n = 0. \tag{7.31}$$

With the estimates (7.28) and (7.29) in place, Theorem 7.7 implies that for any time t,

$$\operatorname{osc}_{M\times\{t\}} v \leq C. \tag{7.32}$$

Estimates (7.28), (7.31) and (7.32) together imply that for all t,

$$\sup_{M\times\{t\}} \left[|v| + \left| \frac{\partial}{\partial t} v \right| \right] \leq C. \tag{7.33}$$

Now let

$$\Phi = \log \operatorname{tr}_{g_+} h_+ - \lambda v.$$

Arguing as in the proof of Theorem 1.2, and using (7.33) yields

$$\left(\frac{\partial}{\partial t} - \Delta \right) \Phi \leq C - C^{-1} e^{-C+\Phi}.$$

A direct application of the maximum principle then yields a uniform upper bound for Φ, which after manipulations yields

$$g_+^u \geq C^{-1} h_+. \tag{7.34}$$

Combining (7.27), (7.29), and (7.34) yields

$$C^{-1} h \leq g_t^u \leq C h.$$

We invoke ([40] Theorem 1.2) to obtain uniform C^∞ estimates for g_t for all times.

We now establish exponential C^∞ convergence of the flow. This follows from an argument of Li-Yau type [27], which we only sketch, as it follows standard lines. At this point we have established uniform upper and lower bounds as well as uniform estimates on all space and time derivatives for the metrics. Now let f_t denote a positive solution to the time-dependent heat equation $(\frac{\partial}{\partial t} - \Delta_{g_t}) f = 0$. A lengthy series of estimates shows that there exists $\alpha > 0$ and a constant C such that

$$|\nabla f|^2 - \alpha f_t \leq C \left(1 + \frac{1}{t} \right).$$

Integrating this over paths in spacetime yields the Harnack estimate

$$\sup_M u(x, t_1) \leq \inf_M u(x, t_2) \left(\frac{t_2}{t_1} \right)^C \exp \left(\frac{C}{t_2 - t_1} + C(t_2 - t_1) \right).$$

Now for $n \in \mathbb{N}$ we define

$$\begin{aligned}
\mu_n(x,t) &= \sup_M f(x, n-1) - f(x, n-1+t),\\
\nu_n(x,t) &= f(x, n-1+t) - \inf_M f(x, n-1),\\
\operatorname{osc}(t) &= \sup_M f(x,t) - \inf_M f(x,t).
\end{aligned}$$

All μ_n and ν_n are solutions to the time-dependent heat equation and so by the Harnack estimate with $t_1 = \frac{1}{2}$, $t_2 = 1$ we obtain

$$\begin{aligned}
\sup_M f(x, n-1) - \inf f(x, n - \tfrac{1}{2}) &\leq C \left(\sup_M f(x, n-1) - \sup_M f(x, n) \right),\\
\sup_M f(x, n - \tfrac{1}{2}) - \inf f(x, n-1) &\leq C \left(\inf_M f(x,n) - \inf_M f(x, n-1) \right).
\end{aligned}$$

Adding these together yields

$$\operatorname{osc}(n-1) + \operatorname{osc}(n - \frac{1}{2}) \leq C(\operatorname{osc}(n-1) - \operatorname{osc}(n)),$$

hence

$$\operatorname{osc}(n) \leq \lambda \operatorname{osc}(n-1),$$

where $\lambda = \frac{C-1}{C} < 1$. Since the oscillation function is nonincreasing it follows that $\operatorname{osc}(t) \leq Ce^{-\lambda t}$. Applying this discussion to $\frac{\partial}{\partial t} u$ shows that it converges exponentially to a constant. Since u and v differ only by time dependent constants, it follows that $\frac{\partial}{\partial t} v$ also converges exponentially to a constant, which must be zero by (7.31). It follows directly that the metric is converging exponentially, and that the limiting metric satisfies $\chi(g_{\pm}^{v_\infty}) = 0$. Lemma 6.3 now implies that g^{v_∞} is Calabi-Yau. The remaining claims of the theorem follow directly from Theorem 7.12. $\square$

Proof of Corollary 1.4. To prove the corollary we simply show that the assumptions imply the setup of Theorem 1.3. Fix Hermitian metrics $\widetilde{h}_{\pm}$ on $T^{1,0}_{\pm}$. Since by assumption $c_1^{BC}(T^{1,0}_{\pm}) = 0$, there exist smooth functions $f_{\pm}$ such that

$$\rho(e^{f_{\pm}} \widetilde{h}_{\pm}) = \rho(\widetilde{h}_{\pm}) - \sqrt{-1}\partial\overline{\partial} f = 0.$$

Now let $h = e^{f_+}\widetilde{h}_+ \oplus e^{f_-}\widetilde{h}_-$. It follows directly that

$$\rho(h) = \rho(h_+) + \rho(h_-) = 0, \qquad \chi(h_{\pm}) = \rho^+(h_+) - \rho^-(h_+) + \rho^-(h_-) - \rho^+(h_-) = 0.$$

Thus $c_1^{BC}(J_A) = 0$, and $\chi(J_A, J_B) = 0$, and so the corollary follows from Theorem 1.3. $\square$

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